

A Tutorial Introduction to the Logic of Parametric Probability

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May, 2012

ABSTRACT

The computational method of parametric probability analysis is introduced. It is demonstrated how to embed logical formulas from the propositional calculus into parametric probability networks, thereby enabling sound reasoning about the probabilities of logical propositions. An alternative direct probability encoding scheme is presented, which allows statements of implication and quantification to be modeled directly as constraints on conditional probabilities. Several example problems are solved, from Johnson-Laird's aces to Smullyan's zombies. Many apparently challenging problems in logic turn out to be simple problems in algebra and computer science: systems of polynomial equations or linear optimization problems. This work extends the mathematical logic and parametric probability methods invented by George Boole.

1 INTRODUCTION

This essay introduces *parametric probability analysis*, a method to compute useful symbolic and numeric results from probability models that contain parameters treated algebraically as unknown variables. A convention is provided to embed formulas from the propositional calculus into such parametric probability models, thereby enabling sound reasoning about the probabilities of logical propositions. An alternative scheme of direct probability encoding is presented, which allows statements of implication and consequence to be modeled directly as constraints on conditional probabilities (without intermediate formulas from the propositional calculus). With direct encoding, probabilities can be used to extend classical logical quantifiers into more precise proportional statements of quantification (or even arbitrary polynomial constraints involving such proportions). Several example problems are analyzed, using the Probability Query Language (PQL) computer program developed by the author. It turns out that many apparently challenging problems in logic and probability are in fact simple problems in algebra and computer science: systems of polynomial equations and inequalities, general search problems, polynomial fractional optimization problems, and in some cases just linear optimization problems.

This work is a continuation of George Boole's pioneering formulation of mathematical logic and probability, codified in his 1854 *Laws of Thought* [3]. It complements the formalization of Hailperin [12] by parsing Boole's notation in a substantially different way that respects Boole's overloaded use of operator signs and numerals. Recognizing Boole's operator overloading, as did Venn many years ago [32], obviates the need to invoke unusual arithmetic or heaps that are not quite sets in order to explain Boole's calculations. There were several innovations in Boole's methodology: the representation of logical formulas and axioms as polynomial formulas and equations; a means to embed logical formulas within probability models; a database-and-query model of interaction; and two-phase inference, with a primary phase of symbolic probability inference followed by a secondary phase of more general algebraic and numerical analysis. The computational method introduced here adds several features to extend Boole's original work: explicit probability-network models; structured probability queries; clearer semantics for embedding propositional-calculus formulas versus directly encoding implication with conditional probabilities; and broadened secondary analysis that includes search and general algebra as well as optimization.

VARIABLE	ROLE	DESCRIPTION	DOMAIN
P	Primary	Proposition: It is a bird	$\{T, F\}$
Q	Primary	Proposition: It can fly	$\{T, F\}$
R	Primary	Value of $(P \rightarrow Q)$	$\{T, F\}$
A	Primary	Value of (3)	$\{3\}$
B	Primary	Number true in $\{P, Q, R\}$	$\{0, 1, 2, 3\}$
C	Primary	Value of $(A - B)$	$\{0, 1, 2, 3\}$
x	Parameter	Probability that P is true	$[0, 1]$
y	Parameter	Probability that Q is true if P is true	$[0, 1]$
z	Parameter	Probability that Q is true if P is false	$[0, 1]$

Table 1 Variables in the probability network \mathcal{M}_{PQ} describing a creature that might be a bird and might be able to fly.

2 PARAMETRIC PROBABILITY NETWORKS

2.1 The Problem with Penguins

Let us begin with a problem that has vexed quite a few philosophers and computer scientists. How can logic be used to reason about an implication that is true sometimes but not always? Following artificial-intelligence tradition [28] we contemplate the problem that most birds can fly but some cannot. To model this problem we shall build a *parametric probability network* denoted \mathcal{M}_{PQ} . This modeling formalism is built from the symbolic algebra used by de Moivre and Bernoulli in their foundational 18th-century treatises on probability [22, 2]; from the parametric treatment of probability developed by Boole in the 19th century [3]; from the axioms of probability theory provided by Kolmogorov in the early 20th century [18]; from the relational databases developed by Codd in the 1970s [8]; and from the Bayesian belief networks and influence diagrams developed in the 1980s by Pearl, Howard, and others [27, 13]. A parametric probability network has four parts: a set of variables, a set of constraints, a network graph, and a set of component probability tables. These parts can be described in a formal, structured language that is suitable for processing by computers as well as by humans.

2.2 Variables and Constraints

First let us introduce two *primary variables* to represent logical propositions about a hypothetical creature: P that it is a bird, and Q that it can fly. For this example each of these primary variables may be either true or false, abbreviated T and F. We would like to consider the truth value of the logical statement ‘ P implies Q ’ and its relation to various probabilities involving P and Q . To facilitate this we add a third primary variable R defined as the value of the formula $P \rightarrow Q$, where the arrow denotes the usual ‘if/then’ material-implication operator of propositional logic [10]. This definition is denoted $R := (P \rightarrow Q)$, with the custom that the expression after the definition sign uses another mathematical system that is *embedded* within the probability model (in this case the propositional calculus). Following Nilsson, one way to describe the truth value of a logical proposition in the context of probability is to use the probability that the proposition is true [23]; let us call this ‘fractional truth value’. Thus $\Pr(P = T)$ describes the fractional truth value of the atomic formula P ; $\Pr(Q = T)$ describes the fractional truth value of the atomic formula Q ; and $\Pr(R = T)$ describes the fractional truth value of the compound formula $P \rightarrow Q$.

Next let us introduce three more variables, x , y , and z , to be used as *parameters* for specifying the probabilities of P and Q . Since parameters are treated differently from primary variables during analysis, we maintain a distinction between these two *roles* that a variable may play. In contrast to P , Q , and R , which share the domain $\{T, F\}$ of two possible values, the parameters x , y , and z may take real-number values between zero and one: thus the domain of each is the interval $[0, 1]$. Finally let us add three more primary variables A , B , and C , with the declarations that B and C share the domain $\{0, 1, 2, 3\}$ of four possible integer values and that A has the set $\{3\}$ of a solitary possible value. We define A as the value of the number 3; B as the number of true propositions in the set $\{P, Q, R\}$; and C as the value of the difference $A - B$. Thus $A := 3$ and $C := (A - B)$, now using integer arithmetic instead of propositional logic as the embedded mathematical system. Table 1 lists all the variables in the parametric probability network \mathcal{M}_{PQ} . Using

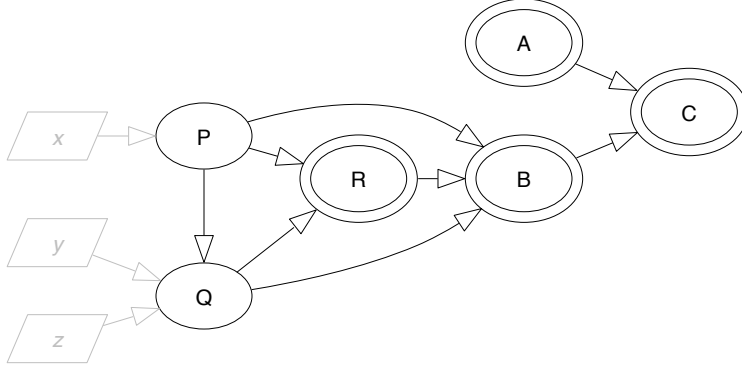


Figure 1 Probability network graph of model \mathcal{M}_{PQ} , with idiosyncratic graphical notation explained in Section 2.3.

m for the number of primary variables and n for the number of parameters, this model has $m = 6$ and $n = 3$.

This document uses the typographical conventions that primary variables are rendered as uppercase italic letters (A, P, Q , etc.) and parameters as lowercase italic letters (x, y , etc.); variable names are case-sensitive, so for example x and X are considered different variables. States of primary variables are rendered in sans-serif type (such as True, False, T, F, 0, 1, 2, 3). These practices are intended to distinguish numbers and formulas in embedded mathematical systems (such as ' $P \rightarrow Q$ ' which is a formula in the propositional calculus) from numbers and formulas in the host probability model (such as ' $1 - p + pq$ ' which is a formula in the algebra of polynomials with rational coefficients). Additionally, different symbols are used to disambiguate different meanings of the equal sign: the colon and equal sign $:=$ for definition or assignment; the double right arrow \Rightarrow for evaluation (as in $2 + 2 \Rightarrow 4$); and the standard equal sign $=$ for the test or assertion of equality. For this presentation, all primary variables are discrete (with finite sets of possible values) and all parameters are continuous (taking rational or real-number values). But in general, parametric probability networks are allowed to have continuous primary variables and discrete parameters too.

2.3 Network Graph and Component Probability Tables

The graph in Figure 1 shows how the variables in the parametric probability network \mathcal{M}_{PQ} are related to one another; it also dictates which component probability tables must be specified in order to complete the model. Following standard probability-network notation, each oval node in this directed acyclic graph represents a primary variable and each directed edge shows a correlation or functional dependence relationship. The absence of an edge is an assertion of independence. In the author's idiosyncratic notation parameters are included in the graph and drawn with parallelogram nodes; furthermore a clique (fully-connected subset of nodes) will be indicated by a small diamond and undirected edges (as illustrated in the examples in Section 7). Double borders on a node indicate that the corresponding variable is *deterministic*: for a primary variable this means that every component probability must be either 0 or 1; for a parameter this means that its value must be fixed at some constant.

The probability of each primary variable must be specified as a function of its parents in the network graph. Thus for the model \mathcal{M}_{PQ} we must provide several component probabilities based on the graph in Figure 1. We must specify the probability of the primary variable P as a function of its parent, the parameter x ; and the conditional probability of Q given its primary-variable parent P as a function of its parameter parents y and z . These component probability distributions appear in Table 2 parts (a) and (b). We must also specify the conditional probability of R given its parents P and Q . For this we transcribe the truth table of the logical formula $P \rightarrow Q$ into the conditional probability table shown as Table 2 part (c). The conditional probabilities for B given P, Q , and R , shown as Table 2 part (d), encode the number of true parent variables. The primary variable A has one possible state which is assigned probability one as shown in Table 2 part (e). Finally, for the primary variable C we specify as Table 2 part (f) a transcription of the table that gives the value of the arithmetical formula $A - B$ using $A = 3$ and integer values of B between 0 and 3. Component probabilities specified by the user are designated $\text{Pr}_0(\dots)$, with the subscript 0 used to distinguish these input values from the output probabilities $\text{Pr}(\dots)$ later computed from them.

(a)

P	$\Pr_0(P)$
T	x
F	$1-x$

(b)

$\Pr_0(Q P)$		
P	$Q = \text{T}$	$Q = \text{F}$
T	y	$1-y$
F	z	$1-z$

(c)

$\Pr_0(R P,Q)$			
P	Q	$R = \text{T}$	$R = \text{F}$
T	T	1	0
T	F	0	1
F	T	1	0
F	F	1	0

(e)

A	$\Pr_0(A)$
3	1

(d)

$\Pr_0(B P,Q,R)$							
P	Q	R	$B = 0$	$B = 1$	$B = 2$	$B = 3$	
T	T	T	0	0	0	1	
T	T	F	0	0	1	0	
T	F	T	0	0	1	0	
T	F	F	0	1	0	0	
F	T	T	0	0	1	0	
F	T	F	0	1	0	0	
F	F	T	0	1	0	0	
F	F	F	1	0	0	0	

(f)

$\Pr_0(C A,B)$					
A	B	$C = 0$	$C = 1$	$C = 2$	$C = 3$
3	0	0	0	0	1
3	1	0	0	1	0
3	2	0	1	0	0
3	3	1	0	0	0

Table 2 Component probability tables for the model \mathcal{M}_{PQ} . The subscript in $\Pr_0(\dots)$ identifies these as user input.

In general the component probability table for a primary variable must contain an element for each of its possible states, given every unique combination of states of its primary-variable parents (an empty set of parents is considered to have one combination of states). Each of these component probabilities must be a polynomial function of the model parameters (with real or rational coefficients). The user may specify arbitrary polynomial equality and inequality constraints on the model parameters; the system adds constraints as needed to enforce the laws of probability (that the feasible values of each component probability must lie between zero and one, and that at every feasible point the probabilities of mutually exclusive and collectively exhaustive events must add up to one). Parameters do not get their own probability distributions; that is precisely how they differ from primary variables. Hence in \mathcal{M}_{PQ} there are no component probability tables for the parameters x , y , and z . However parameters are always subject to algebraic constraints, in this case the zero-one bounds given in Table 1: thus $0 \leq x \leq 1$, $0 \leq y \leq 1$, and $0 \leq z \leq 1$.

2.4 Computable Specification

Here is the computable specification of the parametric probability model \mathcal{M}_{PQ} . The corresponding file was processed by the author's computer program to generate the figures and tables in this section.

```
// basic1.pql: with propositional calculus and integer arithmetic embedded

parameter x { label = "Probability that $$P$ is true"; range = (0,1); }
parameter y { label = "Probability that $$Q$ is true if $$P$ is true"; range = (0,1); }
parameter z { label = "Probability that $$Q$ is true if $$P$ is false"; range = (0,1); }

primary P { label = "Proposition: It is a bird"; states = binary; }
probability ( P ) { data = (x, 1-x); noverify; }

primary Q { label = "Proposition: It can fly"; states = binary; }
probability ( Q | P ) { data = (y, 1-y, z, 1-z); noverify; }

primary R { label = "Value of $(P \rightarrow Q)$"; states = binary; }
probability ( R | P Q ) { function = "R <-> P -> Q ? 1 : 0"; }

primary A { label = "Value of $(3)$"; states = range( 3, 3 ); }
```

```

probability ( A ) { data = (1); }

primary B { label = "Number true in  $\{P,Q,R\}$ "; states = range( 0, 3 ); }
probability ( B | P Q R ) { function = "B == P + Q + R ? 1 : 0"; }

primary C { label = "Value of  $(A - B)$ "; states = range( 0, 3 ); }
probability ( C | A B ) { function = "C == A - B ? 1 : 0"; }

net { graph = 'subgraph { rank=same; "P"; "Q"; }'; } // hint for graph drawing

```

Like the Structured Query Language (SQL) for relational databases [9], the author's Probability Query Language includes a data definition language for specifying models and a data manipulation language for asking queries. The PQL data definition language, which you see illustrated above, has syntax like the ubiquitous C programming language [16] and was also inspired by the NET modeling language from the Hugin system for Bayesian-network inference [14]. The PQL data manipulation language, which is demonstrated below, uses keywords that the user may type as commands to an interactive shell or alternatively include in source files for a batch processor. The command-line shell (called `pqlsh`) and the batch processor (called `pqlpp`) are built atop the Tcl scripting language [25]. There is implicitly a structured language for the *results* of PQL queries as well; these results are essentially relational-database tables whose entries include natural numbers, character strings, and fractional polynomials with rational coefficients.

3 EMBEDDED MATHEMATICAL SYSTEMS

Probability can be used to reason about formulas that are governed by other mathematical systems such as the propositional calculus or integer arithmetic. Such mathematical formulas can be *embedded* within parametric probability networks using the method described here. There are three main aspects to embedding mathematical formulas in probability networks: assigning prior probabilities to the probability-network variables copied from the mathematical variables; assigning conditional probabilities to the probability-network variables created to represent the non-atomic mathematical formulas; and modeling unknown mathematical functions.

3.1 Prior Probabilities for Propositional Variables

To begin the embedding process we copy the variables used in the mathematical formulas of interest into primary variables in the parametric probability network; then we assign these primary variables a prior probability distribution. Let us take the simple view that a mathematical formula ϕ is a finite-length string that may contain only constants, variables, operator signs, and parentheses; a set of grammatical rules dictates which strings constitute well-formed formulas. The constants must be members of some set K of elementary values, and each variable P_1, P_2, \dots, P_v ranges over values in this set K . When each variable in a formula is assigned a constant value from K , the formula itself must also have a value in K ; thus we imagine the set K to be the domain of a mathematical structure that is closed under the allowed operations. For this discussion, we assume that each formula ϕ is finite in length and that the number of formulas under consideration is finite; hence the number v of primary variables is also finite. Let us further limit our attention to the case that the set K of elementary values is finite, with some size d ; this is adequate for modeling 'logical' systems with limited numbers of elementary truth values (for example the usual two).

UNINFORMATIVE PRIORS, PARAMETRICALLY The simplest way to specify a prior probability distribution on the variables P_1 through P_v is to avoid independence assertions and to specify directly the joint probability distribution $\text{Pr}_0(P_1, P_2, \dots, P_v)$. Recall that the subscript 0 in $\text{Pr}_0(\dots)$ designates component probabilities input by the user, as opposed to computed probabilities $\text{Pr}(\dots)$ output by the system. Anyway in this joint-prior case the corresponding probability-network graph has the variables P_1 through P_v joined as a clique. With d elementary values in the set K it is necessary to provide d^v individual probabilities in order to specify completely the distribution $\text{Pr}_0(P_1, P_2, \dots, P_v)$. The laws of probability require that each probability must lie between zero and one, and that the sum of the probabilities in this joint distribution must be one (since they describe mutually exclusive and collectively exhaustive events). To provide a truly *uninformative* prior probability distribution, we should use parameters to state exactly these properties

and nothing more. For example, we might specify the respective probabilities as the variables x_1, x_2, \dots, x_{d^v} subject to the constraints that each $x_i \geq 0$, each $x_i \leq 1$, and the sum $\sum_i x_i = 1$.

It is also possible to specify the joint distribution $\text{Pr}_0(P_1, P_2, \dots, P_v)$ indirectly by using a probability-network subgraph with directed edges; if this subgraph is fully-connected then it still does not introduce independence assertions. For example we might specify the unconditioned probability distribution $\text{Pr}_0(P_1)$ and then the conditional probability distributions $\text{Pr}_0(P_2|P_1)$, $\text{Pr}_0(P_3|P_1, P_2)$, and so on until $\text{Pr}_0(P_v|P_1, P_2, \dots, P_{v-1})$. De Moivre used exactly this construction centuries ago to describe the joint probability of several dependent events [22]. In general many different fully-connected network graphs are possible; each requires a particular set of probabilities to be supplied by the user.

JUST ADD INFORMATION There are several ways to add information beyond uninformative parametric prior probabilities, should the user wish to do so: through the choice of values in the component probability tables; through explicit algebraic constraints on the parameters used to specify component probabilities; and through independence assertions (modeled as the absence of arcs in the probability-network graph) which indirectly provide algebraic constraints. For example, the user may desire to specify that the primary variables P_1, P_2, \dots, P_v are probabilistically independent: in this case they are not directly connected in the network graph, and the joint probability $\text{Pr}(X_1 = k_1, X_2 = k_2, \dots, X_v = k_v)$ that each variable X_i takes the elementary value k_i is constrained to equal the product of the concordant individual component probabilities: $\text{Pr}_0(X_1 = k_1) \times \text{Pr}_0(X_2 = k_2) \times \dots \times \text{Pr}_0(X_v = k_v)$. As another example, it may be desirable to constrain each prior probability to be strictly greater than zero, in order to specify that no combination of primary-variable values is considered impossible *a priori*. The tools of parametric probability, including graphical models and algebraic constraints, allow the user to say exactly what he or she means about prior probabilities.

FOR THE BIRDS To illustrate copied mathematical variables and prior probabilities, in the \mathcal{M}_{PQ} model there are two embedded formulas $P \rightarrow Q$ and $A - B$. The first formula uses the propositional calculus, which includes the set $\{T, F\}$ of elementary values representing truth and falsity as well as the operator \rightarrow for material implication (along with the usual operators \wedge, \vee, \neg , etc.). We copy the propositional variables P and Q into the probability network as primary variables, each with the domain $\{T, F\}$. In lieu of the joint distribution $\text{Pr}_0(P, Q)$, for this model we specify the component probabilities indicated by the fully-connected subgraph with a directed arc from P to Q ; hence the required component probabilities are $\text{Pr}_0(P)$ and $\text{Pr}_0(Q|P)$ which appear in Table 2. The constraints on the real-valued parameters x, y , and z used in these component probability tables provide no more information than the laws of probability require. The value of each parameter must lie between 0 and 1. Here algebra takes care of the sum-to-one constraints; for example in $\text{Pr}_0(P)$ it is tautological that $x + (1 - x) = 1$. The second embedded formula $A - B$ uses integer arithmetic, which includes set $\{0, 1, 2, 3\}$ of elementary values (for convenience we focus on this finite subset of \mathbb{Z}) and the operator $-$ for subtraction (along with the usual operators $+, \times$, etc.). The variables A and B are copied into the probability network as primary variables. For the definition $A := 3$ we add the prior information that the only possible state of A is 3; hence the simple component probability table in Table 2 part (e) assigns probability one to this event. The component probability table specified in Table 2 part (d) encodes that the state of B expresses the number of primary variables among P, Q , and R that have the state T.

3.2 Conditional Probabilities for Operations

To continue the embedding process we introduce additional primary variables for the compound formulas of interest. For each non-atomic formula ϕ_i we introduce a new primary variable S_i . The input component probability table for S_i must specify the conditional probability of S_i given the variables $P_{j_1}, P_{j_2}, \dots, P_{j_a}$ used in the formula ϕ_i . This conditional probability table $\text{Pr}_0(S_i|P_{j_1}, P_{j_2}, \dots, P_{j_a})$ must assign probability one to the appropriate value of the formula given each combination of values of its arguments.

Returning to the \mathcal{M}_{PQ} model, we introduce the primary variable R to represent the value of the compound logical formula $P \rightarrow Q$ and we add the primary variable C for the compound arithmetical formula $A - B$. To complete the definition $R := (P \rightarrow Q)$ the input component probability table $\text{Pr}_0(R|P, Q)$ assigns probability one to the appropriate elementary truth value of the statement of material implication, given each combination of truth values of its arguments.

The conditional probability table is derived in the obvious way from the related logical truth table:

$$\begin{array}{c|c|c} P & Q & P \rightarrow Q \\ \hline T & T & T \\ T & F & F \\ F & T & T \\ F & F & T \end{array} \rightsquigarrow \begin{array}{c|c|c|c} \Pr_0((P \rightarrow Q) | P, Q) & & & \\ \hline P & Q & (P \rightarrow Q) = T & (P \rightarrow Q) = F \\ \hline T & T & 1 & 0 \\ T & F & 0 & 1 \\ F & T & 1 & 0 \\ F & F & 1 & 0 \end{array} \quad (1)$$

For example because the logical formula $T \rightarrow F$ has the value F we have assigned the corresponding conditional probability $\Pr_0((P \rightarrow Q) = F | P = T, Q = F)$ the value 1 and its complement $\Pr_0((P \rightarrow Q) = T | P = T, Q = F)$ the value 0. In Equation 1 the complete formula $P \rightarrow Q$ instead of its abbreviation R appears in the heading of the table $\Pr_0((P \rightarrow Q) | P, Q)$; otherwise this component probability table is the same as $\Pr_0(R | P, Q)$ which appears as Table 2 part (c). The other definition $C := (A - B)$ in the \mathcal{M}_{PQ} model is handled essentially the same way. We consider the portion of the mathematical function table for the arithmetical subtraction operator when its first argument is 3 and its second argument is a member of the set $\{0, 1, 2, 3\}$. The derived component probability table follows:

$$\begin{array}{c|c|c} A & B & A - B \\ \hline 3 & 0 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \\ 3 & 3 & 0 \end{array} \rightsquigarrow \begin{array}{c|c|c|c|c|c} \Pr_0((A - B) | A, B) & & & & & \\ \hline A & B & (A - B) = 0 & (A - B) = 1 & (A - B) = 2 & (A - B) = 3 \\ \hline 3 & 0 & 0 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 0 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 0 & 0 \end{array} \quad (2)$$

The same table appears as Table 2 part (f) labeled with the abbreviation C instead of the full embedded formula $A - B$.

3.3 Known and Unknown Unknowns

The number of elementary truth values in a logical system is independent of the idea of assigning a probability to each possible value (or the idea of considering *sets* of elementary values). For example in ordinary algebra, in order to express the idea that an integer is unknown, we do not imagine that there is an integer called ‘unknown’ in the set \mathbb{Z} . Instead we introduce a symbolic variable and declare that it is constrained to take values in the set of integers, for example Y with $Y \in \mathbb{Z}$. We can expand this set-based declaration $Y \in \mathbb{Z}$ into a parametric probability distribution by further specifying that there is some probability p_k that the variable Y takes each integer value k in the set \mathbb{Z} . In this context, to express perfect ignorance about the value of the integer-valued variable Y , we should admit only that each probability p_k takes a real value between zero and one and that all of the probabilities add up to one: hence we constrain each $p_k \in \mathbb{R}$ with $0 \leq p_k \leq 1$ and the infinite sum $\sum_k p_k = 1$.

Returning to logic, let us consider an embedded system similar to the propositional calculus but with the set $\{\text{True}, \text{False}, \text{Unknown}\}$ of three elementary truth values instead of the usual two. Now, in this 3-valued logic it would be a different thing to say that the value of some variable V is unknown than to say that the value of V is certainly the elementary value called Unknown. For the former assertion (the value of V is unknown) we should start with the set-based declaration $V \in \{\text{True}, \text{False}, \text{Unknown}\}$ and optionally expand this declaration into a parametric probability distribution with only the essential probability constraints, such as:

$$\begin{array}{c|c} V & \Pr_0(V) \\ \hline \text{True} & x_1 \\ \text{False} & x_2 \\ \text{Unknown} & x_3 \end{array} \quad x_i \in \mathbb{R}, \quad 0 \leq x_i \leq 1, \quad x_1 + x_2 + x_3 = 1 \quad (3)$$

But for the latter assertion (Unknown is the value of V) we should assign probability one to the value named Unknown:

$$\begin{array}{c|c} V & \Pr_0(V) \\ \hline \text{True} & 0 \\ \text{False} & 0 \\ \text{Unknown} & 1 \end{array} \quad (4)$$

Many multivalued logics confuse these two ideas (considering sets of possible elementary values—perhaps with probabilities attached—versus adding new elementary values).

We can extend the courtesy of parametric representation to unknown *functions* as well as to unknown *variables*. For example, returning to 2-valued logic, the following conditional probability table and constraints describe an unknown binary operation R^* whose arguments P and Q take values in $\{T, F\}$:

$\Pr_0(R^* P, Q)$			
P	Q	$R^* = T$	$R^* = F$
T	T	t_1	$1 - t_1$
T	F	t_2	$1 - t_2$
F	T	t_3	$1 - t_3$
F	F	t_4	$1 - t_4$

$$t_i \in \mathbb{R}, \quad t_i \in \{0, 1\} \quad (5)$$

This parametric probability table encodes $2^4 \Rightarrow 16$ possible logical functions. For example, the material-implication function $R_{1011}^* := (P \rightarrow Q)$ corresponds to the vector $(1, 0, 1, 1)$ of values for the parameters (t_1, t_2, t_3, t_4) ; the biconditional $R_{1001}^* := (P \leftrightarrow Q)$ corresponds to the vector $(1, 0, 0, 1)$; and the always-false function $R_{0000}^* := F$ corresponds to the vector $(0, 0, 0, 0)$.

4 PRIMARY ANALYSIS: SYMBOLIC PROBABILITY INFERENCE

Having defined the parametric probability network \mathcal{M}_{PQ} with embedded formulas from the propositional calculus and from integer arithmetic, let us now proceed with some analysis. We follow the ingenious framework laid out by Boole in his *Laws of Thought* [3], which included a database-and-query model of interaction between the user and the analytic system, as well as a two-phase model of inference. Boole considered a parametric probability model (the *data* in his terminology), to which a probability query could be posed (his *quaesitum*). In the first phase of analysis, a polynomial formula (Boole’s *final logical equation*) was calculated to answer the query; the variables in this polynomial were the unknown parameters in the probability model. In the second phase of analysis, the minimum and maximum feasible values of this polynomial formula were computed (Boole’s *limits*), subject to constraints that expressed the laws of probability. Parametric probability analysis follows Boole’s two-phase model of inference: in the primary phase of analysis, parametric probability networks and structured probability queries are processed by *symbolic probability-network inference* to compute polynomial solutions; and in the secondary phase of analysis, these polynomial solutions are used for additional algebraic and numerical analysis.

4.1 Structured Probability Queries

Each probability-table query asks for the probability distribution of a *principal* set of primary variables, given some *conditioning* set of primary variables; the remaining primary variables are considered the *marginal* set. Any of these three sets may be empty, but every primary variable must be assigned to one of these positions to make a valid query. Therefore for a parametric probability network with m primary variables, there are 3^m possible probability-table queries (thus $3^6 \Rightarrow 729$ possibilities for the \mathcal{M}_{PQ} model). The result of a probability-table query is a probability table with one or more elements. Each element in this result table is a polynomial function of the component probabilities or a quotient of such polynomials; as a special case an element can be a plain rational number.

For example, to ask for the fractional truth value of the proposition $P \rightarrow Q$, which is represented in the model \mathcal{M}_{PQ} as the variable R , we ask the probability-table query $\Pr(R)$. This query has the principal set $\{R\}$; its conditioning set is empty; its marginal set $\{P, Q, A, B, C\}$ contains the remaining primary variables. The following session shows how to use the pqlsh command-line interface to load the probability model and to issue this probability-table query; here pqlsh> is the system prompt, user input is shown in *italic type*, and input follows the syntax of the Tcl scripting language [25]:

```
pqlsh> set m [pql::load basic1.pql t]; return;

pqlsh> set tr [$m table R]; $tr infer; $tr print -index
```


Index	R	Pr({R})
1	T	$1 - x + x*y$
2	F	$x - x*y$

Such results can also be generated in typeset form: PQL commands included as special comments in \LaTeX documents are replaced by the `pqlpp` preprocessor with the appropriate query results (that is how this document was generated). Anyway, inspecting the results above and recalling the parameter definitions in Table 1, you can see that the fractional truth value of the formula $P \rightarrow Q$ is a simple polynomial function of the parameters x and y , where x is the input probability that P is true and y is the input conditional probability that Q is true given that P is true. The bound constraints $x \in [0, 1]$ and $y \in [0, 1]$ given in Table 1 are considered to be part of the result (as would be additional parameter constraints if there were any): thus the result $\text{Pr}(R)$ is the above probability table along with the constraints that the parameters x and y take real values between zero and one.

Inspecting the first element $1 - x + xy$ in the result of the table query above, note that $\text{Pr}(R = T) = 0$ if and only if $x = 1$ and $y = 0$ (considering also the preexisting constraints $0 \leq x \leq 1$ and $0 \leq y \leq 1$). In terms of the example, the statement $P \rightarrow Q$ that bird implies flight is certainly false exactly when the creature is certainly a bird ($x = 1$) and it is certain that no bird can fly ($y = 0$). Conversely, from either constraint $x = 0$ or $(x, y) = (1, 1)$ it follows by simple algebra that $\text{Pr}(R = T) = 1$. In other words, the implication $P \rightarrow Q$ that bird implies flight is certainly true if the creature is certainly not a bird ($x = 0$); or if the creature is certainly a bird and it is certain that a bird can fly ($x = 1$ and $y = 1$). Other values of x and y give intermediate values of $\text{Pr}(R = T)$ and its complement $\text{Pr}(R = F)$.

Moving on, in order to ask the conditional probability of Q given P we use the probability-table query $\text{Pr}(Q|P)$:

```
pqlsh> set tqp [[$m table Q | P] infer]; $tqp print -index
```

Index	P	Q	Pr({Q} {P})
1	T	T	$(x*y) / (x)$
2	T	F	$(x - x*y) / (x)$
3	F	T	$(z - x*z) / (1 - x)$
4	F	F	$(1 - x - z + x*z) / (1 - x)$

In this case the principal set is $\{Q\}$, the conditioning set is $\{P\}$, and the marginal set is $\{R, A, B, C\}$. Again the constraints that the values of x , y , and z must lie between 0 and 1 are considered part of the result.

4.2 Simple Table-Based Probability Inference

Symbolic probability-network inference follows principles that were already well-described by de Moivre in the early 18th century (notably before the famous paper from Bayes; in fact Bayes referred explicitly to de Moivre's work) [22, 1]. Modern probability-network inference methods focus on efficiency through clever factoring strategies and sophisticated graph manipulations [20, 21]. But for our purposes, inelegant brute-force inference will suffice; this simple approach also helps to elucidate the polynomial form of the probabilities that are computed. Moreover this simple inference method offers its own routes to improved performance: it happens that the necessary arithmetical operations can be performed in parallel (taking advantage of computers with multiple processors), and that the collected calculations are essentially relational-database operations (for which well-optimized software has been developed).

Simple table-based probability inference requires three steps: joining, aggregation, and division. In the first step the component probability tables are *joined* into the full-joint probability table, by multiplying the component probabilities in a certain fashion. Each full-joint probability is the product of one element from every component probability table; the overall calculation is equivalent to a relational database join operation with a small amount of post-processing. In the second step the full-joint probabilities are *aggregated* to compute the marginal probabilities of the queried events, by adding suitable joint probabilities together. This aggregation step corresponds to the relational database operation of the same name. For a conditional query, one aggregation must be performed for the numerator events and a second aggregation for the denominator events in the result table. The numerator events use the set union of

primary variables in the query's principal and conditioning sets; the denominator events use only the variables in the conditioning set. In the third step the aggregate probability of each numerator event is *divided* by the aggregate probability of its corresponding denominator event; this is also a modified relational join operation. It emerges that each element of the result table is the *quotient of sums of products* of the original component probabilities: in other words a fractional polynomial in the model parameters, if each input component probability was itself a polynomial. When a query's conditioning set is empty, no denominator table is constructed and no division is performed; hence unconditional probability-table queries yield sums of products of component probabilities (thus simple polynomial outputs from simple polynomial inputs).

To illustrate symbolic probability-network inference, consider the conditional probability query $\Pr(Q|P,R)$ for the model \mathcal{M}_{PQ} . The principal set is $\{Q\}$ and the conditioning set is $\{P,R\}$. In order to evaluate this query using simple table-based inference we must first compute the full-joint probability table $\Pr(P,Q,R,A,B,C)$; then compute the numerator table $\Pr(P,R,Q)$ (which uses the set union $\{Q\} \cup \{P,R\}$ of variables from the principal and conditioning sets) and the denominator table $\Pr(P,R)$; and finally divide the corresponding numerator and denominator elements by one another. The number of elements in the full-joint table $\Pr(P,Q,R,A,B,C)$ is given by the product of the number of states for each variable: in this case $2 \times 2 \times 2 \times 1 \times 4 \times 4$ which is 128. Every element in the full-joint probability distribution is the product of several component probabilities. For example, the full-joint probability:

$$\Pr(P = F, Q = F, R = T, A = 3, B = 1, C = 2) \quad (6)$$

is given by the product of the corresponding component probabilities:

$$\begin{aligned} & \Pr_0(P = F) \times \Pr_0(Q = F|P = F) \times \Pr_0(R = T|P = F, Q = F) \times \\ & \Pr_0(A = 3) \times \Pr_0(B = 1|P = F, Q = F, R = T) \times \Pr_0(C = 2|A = 3, B = 1) \end{aligned} \quad (7)$$

Substituting the values from Table 2, this product becomes $(1-x)(1-z)(1)(1)(1)(1)$ which simplifies to $1-x-z+xz$. For the \mathcal{M}_{PQ} model it happens that only 4 of the 128 full-joint probabilities are not zero:

INDEX	P	Q	R	A	B	C	$\Pr(P,Q,R,A,B,C)$
13	T	T	T	3	3	0	xy
55	T	F	F	3	1	2	$x-xy$
74	F	T	T	3	2	1	$z-xz$
103	F	F	T	3	1	2	$1-x-z+xz$

(8)

Here the index numbers are relative to all 128 full-joint probabilities, arranged in a particular lexicographic order.

The numerator table $\Pr(P,R,Q)$ and the denominator table $\Pr(P,R)$ are computed by adding appropriate elements of this full-joint probability table. For example the marginal probability $\Pr(P = F, R = T)$ is given by the sum of the probabilities of the corresponding nonzero elements of the full-joint probability table (in rows 74 and 103):

$$\Pr(P = F, Q = T, R = T, A = 3, B = 2, C = 1) + \Pr(P = F, Q = F, R = T, A = 3, B = 1, C = 2) \quad (9)$$

Substituting the polynomial values from Equation 8 yields:

$$\Pr(P = F, R = T) \Rightarrow (z-xz) + (1-x-z+xz) \Rightarrow 1-x \quad (10)$$

The complete tables for $\Pr(P,R,Q)$ and $\Pr(P,R)$ are shown here:

INDEX	P	R	Q	$\Pr(P,R,Q)$
1	T	T	T	xy
2	T	T	F	0
3	T	F	T	0
4	T	F	F	$x-xy$
5	F	T	T	$z-xz$
6	F	T	F	$1-x-z+xz$
7	F	F	T	0
8	F	F	F	0

INDEX	P	R	$\Pr(P,R)$
1	T	T	xy
2	T	F	$x-xy$
3	F	T	$1-x$
4	F	F	0

(11)

Dividing each element of the numerator $\Pr(P, R, Q)$ result table by the matching element of the denominator $\Pr(P, R)$ result table yields the queried conditional probability table $\Pr(Q|P, R)$:

INDEX	P	R	Q	$\Pr(Q P, R)$
1	T	T	T	$(xy)/(xy)$
2	T	T	F	$(0)/(xy)$
3	T	F	T	$(0)/(x-xy)$
4	T	F	F	$(x-xy)/(x-xy)$
5	F	T	T	$(z-xz)/(1-x)$
6	F	T	F	$(1-x-z+xz)/(1-x)$
7	F	F	T	$(0)/(0)$
8	F	F	F	$(0)/(0)$

(12)

Note that the condition $\Pr(P = F, R = F)$ is impossible; in terms of the embedded logical formulas the material implication $R := (P \rightarrow Q)$ cannot be false if its premise P is false. Hence both probabilities $\Pr(Q = T | P = F, R = F)$ and $\Pr(Q = F | P = F, R = F)$ conditioned on this impossible event involve division by zero. By default such exceptional elements are not displayed; they are included above as the quotient-expression $0/0$ to clarify the calculations that have occurred. Omitting these indeterminate elements and pivoting the table to show the probabilities given each possible condition in the same row, the same result for the query $\Pr(Q|P, R)$ is displayed as:

$\Pr(Q P, R)$				
INDEX	P	R	$Q = T$	$Q = F$
1, 2	T	T	$(xy)/(xy)$	$(0)/(xy)$
3, 4	T	F	$(0)/(x-xy)$	$(x-xy)/(x-xy)$
5, 6	F	T	$(z-xz)/(1-x)$	$(1-x-z+xz)/(1-x)$

(13)

4.3 Linear Functions Follow Form

If the primary variables P_1, P_2, \dots, P_v representing embedded variables are modeled as a clique with a single parameter specifying each probability in the joint distribution $\Pr_0(P_1, P_2, \dots, P_v)$, and furthermore if there are no other parameters in the probability network, then all inferred probabilities must be either linear functions of the parameters or quotients of such linear functions. This special case turns out to be quite useful, as it is the natural way to model many problems combining logic and probability. For example, if we were to modify the \mathcal{M}_{PQ} model so that the prior probabilities on P and Q were specified as this joint distribution $\Pr_0(P, Q)$ instead of as the separate probabilities $\Pr_0(P)$ and $\Pr_0(Q|P)$:

P	Q	$\Pr_0(P, Q)$
T	T	x_1
T	F	x_2
F	T	x_3
F	F	x_4

(14)

then all probabilities inferred from this revised model would be linear functions of its parameters or quotients of such linear functions. For example:

P	R	Q	$\Pr(P, R, Q)$
T	T	T	x_1
T	T	F	0
T	F	T	0
T	F	F	x_2
F	T	T	x_3
F	T	F	x_4
F	F	T	0
F	F	F	0

P	R	$\Pr(P, R)$
T	T	x_1
T	F	x_2
F	T	$x_3 + x_4$
F	F	0

$\Pr(Q P, R)$			
P	R	$Q = T$	$Q = F$
T	T	$(x_1)/(x_1)$	$(0)/(x_1)$
T	F	$(0)/(x_2)$	$(x_2)/(x_2)$
F	T	$(x_3)/(x_3 + x_4)$	$(x_4)/(x_3 + x_4)$

(15)

4.4 Handling Division by Zero

You may have noticed that quotients such as $(xy)/(xy)$ and $(z - xz)/(1 - x)$ were not simplified in the results displayed above. That is because it is important to recognize when division by zero is possible: this corresponds to the imposition of an impossible condition, in which case conditional probabilities are appropriately undefined. (Having established that there are no angels dancing on the point of a needle, there is no unique answer to what proportion are boy-angels versus girl-angels.) Let us consider two additional options for handling division by zero within the framework of parametric probability.

DOUBLE-BACKSLASH NOTATION It is useful to have more compact notation to describe the value of a quotient whose denominator might be zero. For this the following convention is proposed: let us say that the value of an expression $\gamma \parallel \alpha = \beta$ is usually γ , except that if the condition $\alpha = \beta$ holds then the value of the expression is undefined. The double backslash \parallel may be read ‘unless’ or ‘except that it is undefined if’; you may think of it as a distant relative of the set difference operator \setminus . This new construction is similar to the ternary conditional operator $?:$ in the C programming language: in C the expression $a == b ? c : d$ has the value c if the condition $a == b$ holds and d otherwise [16].

Using this double-backslash notation the quotient xy/x would be rendered $y \parallel x = 0$, the quotient x/x would be rendered $1 \parallel x = 0$, and the quotient $0/x$ would be rendered $0 \parallel x = 0$. It is best to avoid double-backslash notation when the undefining condition is tautological: thus $0/0$ should be displayed as such (or some other designation for a value that is *always* undefined) instead of as the less intuitive $1 \parallel 0 = 0$. By this double-backslash convention we can display the result for the query $\Pr(Q|P,S)$ in the following way (compare with the original table in Equation 12):

INDEX	P	R	Q	$\Pr(Q P,R)$
1	T	T	T	$1 \parallel xy = 0$
2	T	T	F	$0 \parallel xy = 0$
3	T	F	T	$0 \parallel xy = x$
4	T	F	F	$1 \parallel xy = x$
5	F	T	T	$z \parallel x = 1$
6	F	T	F	$(1 - x - z + xz)/(1 - x)$
7	F	F	T	$0/0$
8	F	F	F	$0/0$

(16)

Like most of the tables and figures in this document, this result table was generated automatically by the author’s computer program in response to a structured query. The current version of the program is not very good at factoring polynomials: here it has not figured out that the quotient $(1 - x - z + xz)/(1 - x)$ simplifies to $1 - z \parallel x = 1$. It would be good for a future implementation of parametric probability analysis to be integrated with a general-purpose computer algebra system; for temporal reasons this was not done in the current version of the computer program.

ALTERNATIVE PARAMETRIC INDETERMINACY It may be desirable to handle division by zero in a different way, by introducing additional parameters to encode indeterminate values while preserving the semantics that probabilities are proportions that add up to one. For this example we might report the value of $\Pr(Q = T|P = F, R = F)$ as θ and the value of $\Pr(Q = F|P = F, R = F)$ as $1 - \theta$, where θ is a new parameter subject to the constraint $0 \leq \theta \leq 1$ about which no other constraints are allowed. In this way we would maintain the property that these two mutually exclusive and collectively exhaustive conditional probabilities have the sum one and that the value of each probability must lie between zero and one, but we would leave the precise value of each probability indeterminate. For example by this parametric-indeterminacy convention we would consider the sum $(xy)/(xy) + (0)/(xy)$ to have the definite value 1 even if it feasible that $xy = 0$ (in particular when using elements of the result table for the query $\Pr(Q|P,R)$, since in this context these fractional polynomial values would describe the probabilities of mutually exclusive and collectively exhaustive events).

LOGICAL	POLYNOMIAL IN $\mathbb{R}[P, Q]$	POLYNOMIAL IN $\mathbb{F}_2[P, Q]$	DESCRIPTION
T	1	1	Elementary truth
F	0	0	Elementary falsity
$\neg P$	$1 - P$	$1 + P$	Negation (NOT)
$P \wedge Q$	PQ	PQ	Conjunction (AND)
$P \oplus Q$	$P + Q - 2PQ$	$P + Q$	Exclusive disjunction (XOR)
$P \vee Q$	$P + Q - PQ$	$P + Q + PQ$	Inclusive disjunction (OR)
$P \rightarrow Q$	$1 - P + PQ$	$1 + P + PQ$	Material implication
$P \leftrightarrow Q$	$1 - P - Q + 2PQ$	$1 + P + Q$	Biconditional (XNOR)
$P \uparrow Q$	$1 - PQ$	$1 + PQ$	Nonconjunction (NAND)
$P \downarrow Q$	$1 - P - Q + PQ$	$1 + P + Q + PQ$	Nondisjunction (NOR)

Table 3 Boolean representation of logical formulas, illustrated for propositional variables P and Q and for polynomials with real or finite-field coefficients. Using real-number coefficients Boole's 'special law' constraints $P^2 = P$ and $Q^2 = Q$ are necessary to limit the possible values of each variable to $\{0, 1\}$.

4.5 Boolean Polynomials and Coincident Probabilities

Let us briefly review Boole's polynomial notation for logical formulas. The mappings between Boole's polynomials and what is now standard logical notation (our mash-up from Hilbert, Peano, and others) are shown in Table 3. These are sometimes called the 'Stone isomorphisms' after [31]. Despite the convenient phrase 'Boolean translation' it should be noted that Boole did not use polynomials to translate from some other conventional system of symbolic notation for logic—for in his time there was no such convention. (Frege's *Begriffsschrift* was published in 1879, many years after Boole's death in 1864; likewise Boole's lifetime predated the works of Peano and Hilbert in which much of modern logical notation was developed [5].) Nonetheless, when viewed as translation, what Boole described turns out to be a special case of Lagrange polynomial interpolation [6].

Contrary to a very common misrepresentation, Boole's polynomials used ordinary integer coefficients for which $1 + 1 = 2$. However there are some advantages to using instead coefficients in the finite field \mathbb{F}_2 of order 2, which uses integer arithmetic modulo 2 (hence $1 + 1 = 0$, addition and subtraction are the same operation, and each value is its own additive inverse). With coefficients in \mathbb{F}_2 the polynomials that represent logical formulas are simpler in form and the number of distinct polynomials is finite (given a finite set of propositional variables). Table 3 includes mappings from conventional logical formulas to polynomials with coefficients in the binary finite field as well as to polynomials with coefficients in the real numbers.

Boole's polynomial notation for logical formulas is often understood in a monolithic way but it was actually the expression of two different ideas: first, the idea that classical logical operations (conjunction, disjunction, negation, and so on) are equivalent to certain combinations of ordinary arithmetical operations, when elementary truth values are represented as ordinary numbers; and second, the idea that the probability that a logical formula is true is a polynomial function of the probabilities that its constituent propositional variables are true. In a special case the polynomial that denotes a logical formula coincides with the polynomial that expresses the probability that the formula itself is true. It is important to recognize the independence property required for this coincidence, and to generalize a means to compute appropriate probabilities in the case that this independence property does not hold (Boole did both).

COINCIDENCE FROM INDEPENDENT PROPOSITIONAL VARIABLES When the propositional variables in use are modeled as probabilistically independent of one another (as described in Section 3.1), then the probability that any compound formula is true coincides with its Boolean polynomial representation (using real coefficients). For example, as shown in Table 3, the Boolean representation of the logical formula $X \leftrightarrow Y$ is the polynomial $1 - X - Y + 2XY$. This Boolean coincidence principle provides that if X and Y are independent in the probability model, with $\Pr_0(X = T) := x$ and $\Pr_0(Y = T) := y$, then the probability $\Pr((X \leftrightarrow Y) = T)$ that the compound formula $X \leftrightarrow Y$ is true has the value $1 - x - y + 2xy$ that mirrors the Boolean polynomial representation $1 - X - Y + 2XY$ of this compound formula. This coincidence is not generally present when the propositional variables in use are probabilistically correlated. For example if $\Pr_0(X = T) := x$, $\Pr_0(Y = T | X = T) := y$, and $\Pr_0(Y = T | X = F) := z$, then the probability $\Pr((X \leftrightarrow Y) = T)$

has the value $1 - x - z + xy + xz$. This value is different from $1 - x - y + 2xy$ exactly when $y \neq z$, in other words when the probabilities of X and Y are nontrivially correlated. The general symbolic probability-network inference method discussed in this section computes correct answers with or without the assertion that the propositional variables are independent.

5 SECONDARY ANALYSIS: ALGEBRA, OPTIMIZATION, AND SEARCH

The results generated by symbolic probability-network inference are always algebraic functions of the model parameters; more specifically they must be polynomials or quotients of polynomials, when the domain of each primary variable is finite and when each input component probability is itself a polynomial. These computed polynomials are ordinary mathematical objects that can be manipulated by ordinary mathematical methods. Boole focused on *optimization* as the secondary analysis that followed his symbolic probability inference. In addition to this very useful technique, we can broaden the scope of secondary analysis to include more general applications of symbolic and numerical analysis to the polynomials generated by symbolic probability-network inference. Here we consider algebra, optimization, and search.

5.1 Algebra with Polynomials

Perhaps the simplest kind of secondary analysis for the fractional polynomials computed by symbolic probability-network inference is elementary algebra: these formulas can be added, subtracted, multiplied, and divided to form new fractional polynomials. For example, consider the difference between the probability that the logical formula $P \rightarrow Q$ holds and the conditional probability that Q is true given that P is true. Since the model \mathcal{M}_{PQ} uses the definition $R := (P \rightarrow Q)$, the requisite difference is $\Pr(R = T) - \Pr(Q = T | P = T)$. To compute this we first select the appropriate elements from the result tables presented in Section 4.1:

```
pqlsh> $tr item 1
```

```
1 - x + x*y
```

```
pqlsh> $tqp item 1
```

```
(x*y) / (x)
```

We then ask `pqlsh` to compute the difference (as a human or computer algebra system could easily do):

```
pqlsh> pql::expr "[$tr item 1] - ([$tqp item 1])"
```

```
(x - x^2 - x*y + x^2*y) / (x)
```

Rewriting this difference with \TeX formatting and the double-backslash notation from Section 4.4 we have:

$$\Pr(R = T) - \Pr(Q = T | P = T) \Rightarrow 1 - x - y + xy \parallel x = 0 \quad (17)$$

It follows from Equation 17 that the two quantities $\Pr(R = T)$ and $\Pr(Q = T | P = T)$ are different unless $x = 1$ or $y = 1$ or both (with the caveat that the difference is undefined when $x = 0$). Perhaps this is clearer when the polynomial difference in Equation 17 is factored as $(1 - x)(1 - y)(x/x)$, which plainly has roots $x = 1$ and $y = 1$. In terms of the example, the probability that the statement ‘bird implies flight’ is true differs from the conditional probability of flight given bird, unless the creature is certainly a bird ($x = 1$) or it is certain that all birds can fly ($y = 1$) or both. However the difference is undefined if the creature could not possibly be a bird ($x = 0$), because in that case the requested condition $P = T$ that the creature is a bird is impossible.

As another example of secondary analysis by algebra, let us calculate the expectation $\mathbb{E}(B)$: the mean value of the number B of True propositions among $\{P, Q, R\}$. For this we begin with the result for the probability-table query $\Pr(B)$, computed as described in Section 4.2:

INDEX	B	$\Pr(B)$
1	0	0
2	1	$1 - z - xy + xz$
3	2	$z - xz$
4	3	xy

(18)

The expected value $\mathbb{E}(B)$ is then computed in the usual way as the sum of each possible value $b_i \in \{0, 1, 2, 3\}$ of B weighted by the probability that it is attained: $\sum_i [b_i \times \Pr(B = b_i)]$. Thus we calculate:

$$0 \times (0) + 1 \times (1 - z - xy + xz) + 2 \times (z - xz) + 3 \times (xy) \Rightarrow 1 + z + 2xy - xz \quad (19)$$

which establishes using simple algebra that the expected value $\mathbb{E}(B)$ is the polynomial $1 + z + 2xy - xz$.

5.2 Polynomial Optimization

Next we consider secondary analysis using optimization (mathematical programming). The polynomials computed by symbolic probability-network inference, and the additional polynomials derived by algebraic calculation, can be used as constraints and objective functions in optimization problems. In *Laws of Thought*, Boole routinely sought to calculate the minimum and maximum feasible values of some polynomial objective computed by symbolic probability-network inference, subject to constraints reflecting the laws of probability. However, Boole's 19th-century optimization methods were not very robust. Polynomial optimization problems are still challenging to solve; they are generally nonlinear and nonconvex, and therefore they can have local solutions that are not global solutions. But there are much better global optimization algorithms now, including reformulation-linearization and semidefinite programming techniques [29, 19]. Specifically for the application of parametric probability analysis, the author has developed a new reformulation and linearization algorithm that computes interval bounds on the global solutions to multivariate polynomial optimization problems, using mixed integer-linear program approximations; the user controls the tightness of the bounds and the time required to compute them by setting the number of reformulation variables [24].

To illustrate optimization as secondary analysis, let us consider the minimum and maximum feasible values of the expectation $\mathbb{E}(B)$ given in Equation 19, with the added constraint that there a 75% chance or less that the statement that 'bird implies flight' is true: $\Pr(R = T) \leq 0.75$. Recalling the specification of the \mathcal{M}_{PQ} probability model given in Tables 1 and 2, we include the constraints $0 \leq x \leq 1$, $0 \leq y \leq 1$, and $0 \leq z \leq 1$. For the additional constraint regarding $\Pr(R = T)$ we recall from Section 4.1 that $\Pr(R = T)$ evaluates to $1 - x + xy$. The resulting inequality $1 - x + xy \leq 0.75$ simplifies to $0.25 + xy \leq x$. Hence to find minimum and maximum bounds on $\mathbb{E}(B)$ we must solve the paired polynomial optimization problems:

$$\begin{array}{ll}
\text{minimize :} & 1 + z + 2xy - xz \\
\text{subject to :} & 0.25 + xy \leq x \\
& \text{and : } 0 \leq x \leq 1 \\
& \quad 0 \leq y \leq 1 \\
& \quad 0 \leq z \leq 1
\end{array}
\qquad
\begin{array}{ll}
\text{maximize :} & 1 + z + 2xy - xz \\
\text{subject to :} & 0.25 + xy \leq x \\
& \text{and : } 0 \leq x \leq 1 \\
& \quad 0 \leq y \leq 1 \\
& \quad 0 \leq z \leq 1
\end{array}
\quad (20)$$

Using a small number of reformulation variables, the author's bounded global polynomial optimization solver computes that the global minimum lies in the interval $[0.938, 1.000]$ and that the global maximum lies in the interval $[2.500, 2.594]$. With more reformulation variables the solver generates tighter intervals $[1.000, 1.000]$ and $[2.500, 2.500]$. The solver reports that the global minimum 1.000 is achieved at the point $(x = 0.984, y = 0.000, z = 0.000)$ and that the global maximum 2.500 is achieved at the point $(x = 1.000, y = 0.750, z = 0.266)$. You may notice by inspection of Equation 19 that, absent any constraints besides the bounds on the parameters x , y , and z , the global minimum of the expected value $\mathbb{E}(B)$ is exactly 1 (when $x = y = z = 0$) and its global maximum is exactly 3 (when $x = y = z = 1$). The constraint $\Pr(R = T) \leq 0.75$ added for this example has rendered some of that range infeasible.

INDEX	t_1	t_2	t_3	t_4	$\Pr(B^* = 0)$
1	0	0	0	0	$1 - x - z + xz$
2	0	0	0	1	0
3	0	0	1	0	$1 - x - z + xz$
4	0	0	1	1	0
5	0	1	0	0	$1 - x - z + xz$
6	0	1	0	1	0
7	0	1	1	0	$1 - x - z + xz$
8	0	1	1	1	0
9	1	0	0	0	$1 - x - z + xz$
10	1	0	0	1	0
11	1	0	1	0	$1 - x - z + xz$
12	1	0	1	1	0
13	1	1	0	0	$1 - x - z + xz$
14	1	1	0	1	0
15	1	1	1	0	$1 - x - z + xz$
16	1	1	1	1	0

INDEX	t_1	t_2	t_3	t_4	$\Pr(B^* = 2)$
1	0	0	0	0	xy
2	0	0	0	1	xy
3	0	0	1	0	$z + xy - xz$
4	0	0	1	1	$z + xy - xz$
5	0	1	0	0	x
6	0	1	0	1	x
7	0	1	1	0	$x + z - xz$
8	0	1	1	1	$x + z - xz$
9	1	0	0	0	0
10	1	0	0	1	0
11	1	0	1	0	$z - xz$
12	1	0	1	1	$z - xz$
13	1	1	0	0	$x - xy$
14	1	1	0	1	$x - xy$
15	1	1	1	0	$x + z - xy - xz$
16	1	1	1	1	$x + z - xy - xz$

Table 4 Instantiations of $\Pr(B^* = 0)$ and $\Pr(B^* = 2)$ calculated by substituting the listed values of (t_1, t_2, t_3, t_4) into the corresponding polynomials from Equation 21. In row 10 both instantiated polynomials are identically zero.

5.3 General Search

Providing yet another mode of secondary analysis, polynomials generated by symbolic probability-network inference can be used in general computer-science search problems that might be awkward to formulate in terms of algebraic equations or numeric optimization. To illustrate, note that in the model \mathcal{M}_{PQ} it is impossible for all three propositions P , Q , and R to be false; as Equation 11 shows, the joint probability $\Pr(P = F, R = F, Q = F)$ evaluates to zero. This makes semantic sense based on the definition $R := (P \rightarrow Q)$, for if the premise P is false then the material implication $P \rightarrow Q$ must be true; hence both propositions cannot be false simultaneously. Let us search for a logical proposition that has a different property: a logical function R^* of P and Q such that the number of true propositions among $\{P, Q, R^*\}$ must be odd (either 1 or 3).

To set up this search let us replace the original component probability table $\Pr(R|P, Q)$ shown in Table 2 part (c) with the table for $\Pr(R^*|P, Q)$ given in Equation 5 in which the probabilities of each value of R^* given each combination of values for P and Q are encoded by the parameters t_1 , t_2 , t_3 , and t_4 as described in Section 3.3. Using this replacement parametric table and symbolic probability-network inference as in Section 4.2 we compute the probability distribution on the number B^* of true propositions among $\{P, Q, R^*\}$:

B^*	$\Pr(B^*)$
0	$1 - x - z - t_4 + xz + xt_4 + zt_4 - xzt_4$
1	$x + z + t_4 - xy - xz - xt_2 - xt_4 - zt_3 - zt_4 + xyt_2 + xzt_3 + xzt_4$
2	$xy + xt_2 + zt_3 - xyt_1 - xyt_2 - xzt_3$
3	xyt_1

(21)

Now we desire to find the values of (t_1, t_2, t_3, t_4) for which the only possible values of B^* are 1 and 3. In other words, we require that the polynomials $\Pr(B^* = 0)$ and $\Pr(B^* = 2)$ are both identically zero after substituting the selected values of (t_1, t_2, t_3, t_4) . Considering each value $t_i \in \{0, 1\}$ there are $2^4 \Rightarrow 16$ possible values of the vector (t_1, t_2, t_3, t_4) . For this small problem we simply enumerate every possibility and substitute these values into the polynomials in the result table for $\Pr(B^*)$ given in Equation 21. (Of course the point of most search algorithms is to avoid exhaustive enumeration of the search space; we eschew such luxury for now.) In Table 4 the table on the left gives the instantiations of $\Pr(B^* = 0)$ and the table on the right gives the instantiations of $\Pr(B^* = 2)$ at all 16 possible values of (t_1, t_2, t_3, t_4) .

Comparing these tables you can see that only in row 10 are both polynomials identically zero. In every other case it is either possible that $\Pr(B^* = 0)$ is not zero, that $\Pr(B^* = 2)$ is not zero, or that both probabilities are not

zero.¹ Row 10 corresponds to the vector $(t_1, t_2, t_3, t_4) = (1, 0, 0, 1)$ and in turn to the logical formula $R_{1001}^* := (P \leftrightarrow Q)$ that combines P and Q using the biconditional operator (see Section 3.3). Here is the probability distribution on the number B_{1001}^* of true propositions among the set $\{P, Q, R_{1001}^*\}$, which is obtained from substituting the solution values $(t_1, t_2, t_3, t_4) = (1, 0, 0, 1)$ into Equation 21:

B_{1001}^*	$\Pr(B_{1001}^*)$
0	0
1	$1 - xy$
2	0
3	xy

(22)

Thus by applying general search as secondary analysis, we have computed that the number of true propositions among the set $\{P, Q, P \leftrightarrow Q\}$ must be odd; it cannot happen that exactly 0 or 2 of these propositions are true, regardless of the prior probabilities x , y , and z . Moreover, search has demonstrated that the only other formulas R^* with this odd-number property must have the same truth table as $P \leftrightarrow Q$. For a formula with any other truth table there would be feasible values of the parameters (x, y, z) for which $\Pr(B^* = 0) > 0$, $\Pr(B^* = 2) > 0$, or both.

6 ADDITIONAL MODELING ISSUES

Having discussed primary and secondary analysis of parametric probability networks, there are two additional modeling issues to consider. First, there are two different techniques to model conditions during parametric probability analysis: as denominator events in conditional probability queries and as constraints in optimization problems. Second, it is possible to encode some statements about implication directly as conditional probabilities, without the intermediate device of embedded formulas from the propositional calculus. Using such direct probability encoding, constraints on conditional probabilities can express quantification without the need for classical logical quantifiers and with the option to specify more precise fractional values than just ‘some’.

6.1 Subjunctive Conditions, Imperative Constraints

Problems in logic and probability commonly involve *conditions*, and there are two idioms for representing conditions during parametric probability analysis. Using the *subjunctive* idiom, a condition is modeled as the denominator event in a conditional probability-table query. Using the *imperative* idiom, a condition is modeled as an equality constraint in an optimization problem. These alternative formulations have slightly different semantics. In the subjunctive formulation we ask hypothetically what would be the probability of some event, if the stated condition were to hold. In the imperative formulation we assert factually that the stated condition must hold, and then ask what is the probability of some event under this necessary condition. These two idioms yield essentially the same solutions, although they report the exception that the stated condition is impossible in two different ways: in the subjunctive formulation an impossible condition causes division by zero, but in the imperative formulation an impossible condition produces an unsatisfiable system of equations. Both conditioning idioms can be used for parametric probability networks with or without embedded logical formulas.

EXAMPLE: TWO MODES OF MODUS PONENS There are two different ways to express the familiar phenomenon of *modus ponens* with a parametric probability network, and these correspond to two slightly different questions. First we might ask in a *subjunctive* mood: if P and $P \rightarrow Q$ were to be true, what would be the probability that Q is also true? Second we might specify in an *indicative* mood that P and $P \rightarrow Q$ must certainly be true, and then ask the probability that Q is also true. Recall that in model \mathcal{M}_{PQ} the variable R stands for the proposition $P \rightarrow Q$.

For the subjunctive formulation we use a probability-table query to compute $\Pr(Q|P, R)$, the result of which is shown in Equation 12. The first element of this result table gives the desired conditional probability:

$$\Pr(Q = \text{T} | P = \text{T}, R = \text{T}) \Rightarrow (xy) / (xy) \quad (23)$$

¹For this problem it happens that every polynomial in Table 4 that is not identically zero has a value that is strictly greater than zero at some feasible value of the parameters (x, y, z) . In general it would be necessary to solve an optimization problem to verify that a polynomial which is symbolically different from zero indeed has a feasible value greater than zero, taking into account all of the constraints provided.

The value of this quotient is one unless its denominator is zero (that is, $1 \parallel xy = 0$ using the notation of Section 4.4). We interpret this to mean that if both propositions P and $P \rightarrow Q$ were to be true, then Q would also be true with probability 1 (unless $x = 0$ or $y = 0$ or both, in which exceptional cases the condition $\{P = \text{T}, R = \text{T}\}$ would be impossible and thus the requested conditional probability would be the indeterminate expression $0/0$). In other words, if the creature happens to be a bird, and if it happens to be true that bird implies flight, then it would certainly be true that the creature can fly. Except that if it were already known *a priori* that it is impossible for the creature to be a bird, and/or that there is no chance that a bird can fly, then the question of flight assuming the stated conditions would not have a definite answer because the conditions would be impossible.

Alternatively, to use the indicative formulation, we build an optimization query in which we ask the minimum feasible value of the objective $\Pr(Q = \text{T})$ subject to the following constraints that P and R must certainly be true:

$$\Pr(P = \text{T}) = 1 \quad (24)$$

$$\Pr(R = \text{T}) = 1 \quad (25)$$

We specify the constraint on P using the first element of its result table (the output $\Pr(P)$ happens to be the same as the input $\Pr_0(P)$ shown in Table 2) and the constraint on R from the first element of $\Pr(R)$ shown in Section 4.1:

```
pqlsh> set cp "[[ $m1$  table  $P$ ] infer] item 1] == 1"
```

```
x == 1
```

```
pqlsh> set cr "[ $tr$  item 1] == 1"
```

```
1 - x + x*y == 1
```

The second constraint simplifies to $x = xy$. Moving on, we obtain the objective function as the first element of the result table for the query $\Pr(Q)$:

```
pqlsh> set tq [[ $m1$  table  $Q$ ] infer];  $tq$  print;
```

```
Q      | Pr( {Q} )
-----
T      | z + x*y - x*z
F      | 1 - z - x*y + x*z
```

Our optimization query is a request for a polynomial program using this objective function and these constraints:

```
pqlsh> set pq [ $m1$  pprog -min "[ $tq$  item 1]"  $scp$   $scr$ ]; return;
```

This optimization query generates the following polynomial optimization problem:

$$\begin{aligned} \text{minimize : } & z + xy - xz \\ \text{subject to : } & x = 1 \\ & x = xy \\ \text{and : } & 0 \leq x \leq 1 \\ & 0 \leq y \leq 1 \\ & 0 \leq z \leq 1 \end{aligned} \quad (26)$$

Solving this problem gives bounds on the minimum feasible value of $\Pr(Q = \text{T})$ under the constraints $\Pr(P = \text{T}) = 1$ and $\Pr(R = \text{T}) = 1$:

```
pqlsh>  $pq$  solve;  $pq$  solution
```

```
1.000 1.000
```

pqlsh> \$pq point

{x = 1.000} {y = 1.000} {z = 0.000}

This solution to the optimization problem in Equation 26 demonstrates that proposition Q must be true (that is, the variable Q attains the state T with minimum probability in the interval $[1.000, 1.000]$) if P and $P \rightarrow Q$ are constrained to be true. (You can see by inspection that the global solution to Equation 26 is exactly 1.) In other words, if it is certainly true that the creature is a bird, and it is certainly true that bird implies flight, then it is certainly true that the creature can fly. The exceptional cases $x = 0$ and $y = 0$ are handled differently in this formulation: instead of causing division by zero, they identify infeasible points relative to the constraints in Equation 26.

6.2 Direct Probability Encoding

It is possible to model implication and quantification directly in parametric probability networks, without using the classical logical devices of material implication or universal and existential quantifiers. We have already seen the statement of material implication $P \rightarrow Q$ used to model the idea that Q is a necessary consequence of P (with subjunctive and indicative idioms to impose the condition that this statement of material implication is true). We could instead constrain the conditional probability $\Pr(Q = T | P = T)$ to 1 to express the idea that that Q is a necessary consequence of the premise P . Similarly the constraint $\Pr(Q = T | P = T) = 0$ is an alternative way to model the assertion that Q is never a consequence of P .

These equality constraints on conditional probabilities are alternatives to universally-quantified statements such as $\forall \alpha (P(\alpha) \rightarrow Q(\alpha))$ to say that all P are Q ; or $\forall \alpha (P(\alpha) \rightarrow \neg Q(\alpha))$ to say that no P are Q . In a similar fashion, to model the assertion that Q sometimes follows P we could constrain the relevant conditional probability to be strictly greater than zero: $\Pr(Q = T | P = T) > 0$. And to model the assertion that Q sometimes does not follow P we could constrain the relevant conditional probability to be strictly less than one: $\Pr(Q = T | P = T) < 1$. These inequality constraints on conditional probabilities are alternatives to existentially-quantified formulas such as $\exists \alpha (P(\alpha) \rightarrow Q(\alpha))$ or $\exists \alpha (P(\alpha) \wedge Q(\alpha))$ to say that some P are Q ; or $\exists \alpha (P(\alpha) \wedge \neg Q(\alpha))$ to say that some P are not Q . In the framework of parametric probability, quantified variables like α are distinct both from primary variables and from parameters.

Recall from the secondary analysis in Section 5.1 that the conditional probability that Q is true given that P is true is mathematically distinct from the probability that the material-implication statement $P \rightarrow Q$ is true. These correspond to symbolically different polynomials; the arithmetical difference between them depends on the prior probabilities assigned to the events P and Q as reported in Equation 17.

THE OPTION OF EXISTENTIAL IMPORT The constraints introduced to quantify propositions can be specified such that they do or do not have existential import, as the user desires. In general, constraining *input* component probabilities specified by the user can have different effects from constraining *output* probabilities computed by the system; existential import is one of those mutable effects. Elementary algebra helps to clarify the consequences of various polynomial constraints.

To illustrate, here are the input component probability table $\Pr_0(Q|P)$ from Table 2 part (b); the computed table $\Pr(Q|P)$ for this same conditional probability, created as in Section 4.2; and the computed probability table $\Pr(P)$:

$\Pr_0(Q P)$		
P	$Q = T$	$Q = F$
T	y	$1 - y$
F	z	$1 - z$

$\Pr(Q P)$		
P	$Q = T$	$Q = F$
T	$(xy)/(x)$	$(x - xy)/(x)$
F	$(z - xz)/(1 - x)$	$(1 - x - z + xz)/(1 - x)$

P	$\Pr(P)$
T	x
F	$1 - x$

(27)

Now, adding the constraint $y > 0$ that the *input* value $\Pr_0(Q = T | P = T)$ must be strictly greater than zero would specify that Q is sometimes a consequence of P (when P happens to be true) without asserting that P is ever true. For if $x = 0$ then the computed probability $\Pr(P = T)$ would be zero even if $y > 0$. Thus the input-value constraint $\Pr_0(Q = T | P = T) > 0$, meaning $y > 0$, does not affect the values in $\Pr(P)$; it has no existential import. Incidentally, in the case $x = 0$ the inferred conditional probability $\Pr(Q = T | P = T)$, calculated by the laws of probability as the quotient $\Pr(P = T, Q = T) / \Pr(P = T)$, would be indeterminate due to division by zero—regardless of the value y

assigned to the corresponding input conditional probability $\Pr_0(Q = T | P = T)$. For conditional probabilities, it is essential to distinguish between input and output values; what is computed by the system may differ from what was input by the user (particularly regarding denominators that may be zero).

On the other hand, adding the constraint $(xy)/(x) > 0$ that the *output* value $\Pr(Q = T | P = T)$ is strictly greater than zero would indeed carry existential import. Since both parameters x and y are constrained to be nonnegative, by simple algebra this constraint would require the product xy in the numerator to be strictly greater than zero, which would in turn require both $x > 0$ and $y > 0$. Consequently the output value $\Pr(P = T)$, which has the polynomial value x , would also be required to be strictly greater than zero. So the output-value constraint $\Pr(Q = T | P = T) > 0$, meaning $(xy)/(x) > 0$, would also assert $\Pr(P = T) > 0$; it has existential import.

It is always possible to constrain output probabilities; but which input probabilities are available to constrain depends on the graph structure of the probability network. For example the presence of an input table $\Pr_0(Q|P)$ requires that P is a parent of Q in the network graph. If instead P and Q were joined a clique then their joint probability distribution $\Pr_0(P, Q)$ would be specified as the input component probability table; in this case there would be no separate table $\Pr_0(Q|P)$ of input values to be constrained.

As an implementation detail many optimization solvers do not distinguish between strict and weak inequality constraints. To work around this limitation we can use some small positive constant ϵ and weak inequality constraints $p \geq \epsilon$ and $p \leq 1 - \epsilon$ to approximate the strict inequalities $p > 0$ and $p < 1$ for a polynomial of interest p . For example using $\epsilon = 0.1$ the constraint $y \geq 0.1$ approximates the strict inequality $\Pr_0(Q = T | P = T) > 0$. This approximation scheme allows common optimization solvers to compute results that reliably distinguish between probabilities that are exactly zero, those that are exactly one, and those that have some intermediate value.

FRACTIONAL QUANTIFICATION Using probability directly to model quantification offers an important benefit: we are not limited to the classical quantifiers ‘all’ and ‘some’. Instead it is possible to describe and to constrain the precise proportion of cases for which some logical formula is true or false. In other words, besides the constraints $p = 0$, $p = 1$, $p > 0$, and $p < 1$ that the probability p encoding some statement of quantification is equal to zero, equal to one, strictly greater than zero, or strictly less than one, it is possible to specify arbitrary polynomial constraints on p . Thus in addition to statements like ‘all P are Q ’ or ‘some P are not Q ’ we can model such assertions as ‘exactly c percent of P are Q ’ or ‘between a and b percent of P are Q ’ or ‘if there are any P , then twice as many P are Q as P are R ’. In certain cases, the requisite constraints are guaranteed to be linear in the model parameters.

As a philosophical aside, *probability* is best understood as the proportion of some underlying basic measure. There is diversity in what that basic measure can represent: number or cardinality (in which case the proportion is frequency); the absolute weight of subjective belief or of causal propensity (in which case the proportion is subjective probability); monetary value; mass; or some other property. It is essential for the property chosen as a basic measure to be additive across set unions of measured events, which is the quintessential mathematical property of a measure. In the course of contemplating proportional statements of quantification more precise than ‘all’ and ‘some’, and for reasoning with Nilsson-style fractional truth values, it may be worthwhile to clarify what the basic measure is intended to mean.

7 ANALYSIS OF SELECTED PROBLEMS

Now let us apply parametric probability analysis to an assortment of problems from the literature, each of which has been advertised as being difficult or impossible to solve by formal mathematical methods. We shall see that several well-known problems—about card games with logical rules, axioms with uncertainty, counterfactual conditions, and truthful knights and lying knaves—are nothing more than parametric probability problems. Such problems are easily solved by parametric probability analysis; many of them turn out to be linear optimization problems whose constraints and objectives are the solutions to probability queries.

7.1 Johnson-Laird’s Winning Hand

Let us begin with a problem from Johnson-Laird that was also discussed by Bringsjord [15, 4]:

If one of the following assertions is true then so is the other:

1. There is a king in the hand if and only if there is an ace in the hand.
2. There is a king in the hand.

Which is more likely to be in the hand, if either: the king or the ace? Prove that you are correct.

It is worth repeating the challenge that Bringsjord issued with this problem:

I don't even think Bayesian systems can possibly solve logic problems that involve probability. ... I would very much like to see a Bayesian system take this declarative information as input, and yield the correct answer, and a proof that this *is* the answer. I assure you that I will not hold my breath.

Though the computational method presented here is more appropriately called 'Boolean' than 'Bayesian', there is no difficulty in solving logic problems that involve probability using parametric probability analysis. Proof, such as it is, is supplied by elementary algebra. Let us consider three ways to analyze this ace-king problem: using Boolean polynomials to simplify the logical formula involved; using parametric probability analysis with subjunctive conditioning; and using parametric probability analysis with indicative conditioning.

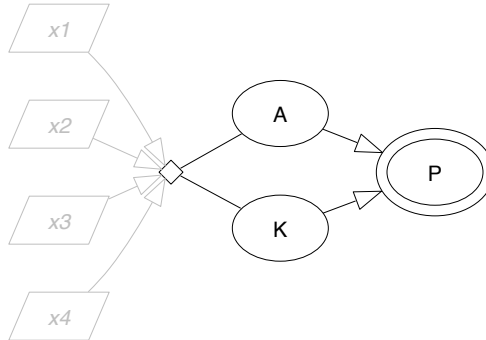
POLYNOMIAL SIMPLIFICATION OF LOGICAL FORMULAS Perhaps the easiest way to solve this ace-king problem is to simplify the logical assertion in it. Using A to represent the proposition that there is an ace in the hand and K to represent the proposition that there is a king in the hand, the problem specifies the assertion $(K \leftrightarrow A) \leftrightarrow K$. This compound logical formula simplifies to the atomic formula A after Boolean polynomial representation using the rules in Table 3. For example using coefficients in the binary finite field \mathbb{F}_2 the inner biconditional translates to the polynomial $1 + K + A$. Substituting this value, the entire formula $(1 + K + A) \leftrightarrow K$ translates to the polynomial $1 + (1 + K + A) + K$. Recall that using integer arithmetic modulo 2 either elementary value 0 or 1 is its own additive inverse; thus all terms in this polynomial $1 + 1 + K + A + K$ cancel out except A . Using real-number coefficients would generate the same answer (keeping in mind that A can be substituted for A^2 and K for K^2 due to Boole's 'special law' constraints $A^2 = A$ and $K^2 = K$). Taking advantage of such polynomial representation and simplification, an equivalent problem statement would be:

There is an ace in the hand.

Which is more likely to be in the hand, if either: the king or the ace?

It is evident that the ace must be equally likely or more likely than the king, since the ace is present with certainty.

PARAMETRIC PROBABILITY MODEL Next let us construct an explicit parametric probability network for this ace-king problem, using the technique described in Section 3. We copy the propositional variables A and K into the probability network as primary variables with the set $\{T, F\}$ of possible values representing elementary truth and falsity. We add a third primary variable P which is defined as the value of the compound logical formula $(K \leftrightarrow A) \leftrightarrow K$ asserted in the problem description. We introduce parameters x_1 through x_4 to specify prior probabilities. Here is the network graph:



(28)

As in Section 3.1 we specify an uninformative prior probability distribution $\Pr_0(A, K)$ on the variables A and K using the x_i parameters, with the constraints $0 \leq x_i \leq 1$ and $x_1 + x_2 + x_3 + x_4 = 1$ to enforce the laws of probability.

Following the method of Section 3.2 we construct a conditional probability table $\text{Pr}_0(P|A, K)$ to express the definition $P := ((K \leftrightarrow A) \leftrightarrow K)$. These two component probability tables complete the ace-king model:

A	K	$\text{Pr}_0(A, K)$	$\text{Pr}_0(P A, K)$		
A	K		P = T	P = F	
T	T	x_1	1	0	(29)
T	F	x_2	1	0	
F	T	x_3	0	1	
F	F	x_4	0	1	

Using this parametric probability network we can compare the probabilities that A and K are true, given the condition that P is true—using both subjunctive and imperative formulations to express the condition.

ANALYSIS IN THE SUBJUNCTIVE MOOD Using the subjunctive idiom for conditioning, we desire the difference between the conditional probabilities $\text{Pr}(A = T | P = T)$ and $\text{Pr}(K = T | P = T)$. The relevant fractional polynomial values are included in the result tables for the queries $\text{Pr}(A|P)$ and $\text{Pr}(K|P)$, computed as described in Section 4.2:

$\text{Pr}(A P)$			$\text{Pr}(K P)$		
P	A = T	A = F	P	K = T	K = F
T	$(x_1 + x_2) / (x_1 + x_2)$	$(0) / (x_1 + x_2)$	T	$(x_1) / (x_1 + x_2)$	$(x_2) / (x_1 + x_2)$
F	$(0) / (x_3 + x_4)$	$(x_3 + x_4) / (x_3 + x_4)$	F	$(x_3) / (x_3 + x_4)$	$(x_4) / (x_3 + x_4)$

The desired difference uses the first element of each table, combined by elementary algebra:

$$(x_1 + x_2) / (x_1 + x_2) - (x_1) / (x_1 + x_2) \Rightarrow (x_2) / (x_1 + x_2) \quad (31)$$

To answer the question posed in the problem we must determine the minimum and maximum feasible values of this difference, subject to the constraints on the parameters involved. These extreme values are the solutions to the following pair of optimization problems, which share a set of linear constraints and a fractional linear objective function:

$$\begin{array}{ll}
\text{minimize :} & (x_2) / (x_1 + x_2) \\
\text{subject to :} & x_1 + x_2 + x_3 + x_4 = 1 \\
& \text{and : } 0 \leq x_1 \leq 1 \\
& \quad 0 \leq x_2 \leq 1 \\
& \quad 0 \leq x_3 \leq 1 \\
& \quad 0 \leq x_4 \leq 1
\end{array}
\quad
\begin{array}{ll}
\text{maximize :} & (x_2) / (x_1 + x_2) \\
\text{subject to :} & x_1 + x_2 + x_3 + x_4 = 1 \\
& \text{and : } 0 \leq x_1 \leq 1 \\
& \quad 0 \leq x_2 \leq 1 \\
& \quad 0 \leq x_3 \leq 1 \\
& \quad 0 \leq x_4 \leq 1
\end{array} \quad (32)$$

One way to solve these fractional linear programs is through the Charnes-Cooper transformation, which converts them into ordinary linear programs [7]. After such reformulation, standard linear optimization finds the minimum value 0 which is achieved at the point $(x_1 = 1, x_2 = 0, x_3 = 0, x_4 = 0)$ and the maximum value 1 which is achieved at the point $(x_1 = 0, x_2 = 1, x_3 = 0, x_4 = 0)$. Thus the difference $\text{Pr}(A = T | P = T) - \text{Pr}(K = T | P = T)$ is bounded by zero and one. This implies the inequality:

$$\text{Pr}(A = T | P = T) \geq \text{Pr}(K = T | P = T) \quad (33)$$

In other words, given the condition $P := ((K \leftrightarrow A) \leftrightarrow K)$ stated in the problem description, it is at least as likely that there is an ace in the hand as a king.

PARAMETRIC PROBABILITY IN THE INDICATIVE MOOD We can find the equivalent solution using the same parametric probability network model, but with an imperative rather than subjunctive query formulation. In this idiom our objective function is the difference $\text{Pr}(A = T) - \text{Pr}(K = T)$ between the unconditional probabilities that there is

an ace versus a king in the hand. The relevant probabilities, computed as in Section 4.2, appear in the results for the probability-table queries $\Pr(A)$ and $\Pr(K)$:

$$\begin{array}{c|c} \hline A & \Pr(A) \\ \hline T & x_1 + x_2 \\ \hline F & x_3 + x_4 \\ \hline \end{array} \quad \begin{array}{c|c} \hline K & \Pr(K) \\ \hline T & x_1 + x_3 \\ \hline F & x_2 + x_4 \\ \hline \end{array} \tag{34}$$

The requisite difference $(x_1 + x_2) - (x_1 + x_3)$ simplifies to $x_2 - x_3$. Continuing on, in this indicative formulation the condition in the problem statement is now modeled as the additional constraint $\Pr(P = T) = 1$, using this result for the query $\Pr(P)$:

$$\begin{array}{c|c} \hline P & \Pr(P) \\ \hline T & x_1 + x_2 \\ \hline F & x_3 + x_4 \\ \hline \end{array} \tag{35}$$

Therefore we formulate the following pair of optimization problems to find the minimum and maximum feasible values of the difference between the probability of the ace and the king, subject to the constraints that the assertion in the problem description is true and that the laws of probability are followed:

$$\begin{array}{ll} \text{minimize :} & x_2 - x_3 \\ \text{subject to :} & x_1 + x_2 + x_3 + x_4 = 1 \\ & x_1 + x_2 = 1 \\ \text{and :} & 0 \leq x_1 \leq 1 \\ & 0 \leq x_2 \leq 1 \\ & 0 \leq x_3 \leq 1 \\ & 0 \leq x_4 \leq 1 \end{array} \quad \begin{array}{ll} \text{maximize :} & x_2 - x_3 \\ \text{subject to :} & x_1 + x_2 + x_3 + x_4 = 1 \\ & x_1 + x_2 = 1 \\ \text{and :} & 0 \leq x_1 \leq 1 \\ & 0 \leq x_2 \leq 1 \\ & 0 \leq x_3 \leq 1 \\ & 0 \leq x_4 \leq 1 \end{array} \tag{36}$$

These are simple linear programs whose objective functions are not fractional. Solving them with standard methods yields the minimum value 0 which is achieved at the point $(x_1 = 1, x_2 = 0, x_3 = 0, x_4 = 0)$ and the maximum value 1 which is achieved at the point $(x_1 = 0, x_2 = 1, x_3 = 0, x_4 = 0)$. In other words, subject to the constraint $\Pr(P = T) = 1$ that the formula $(K \leftrightarrow A) \leftrightarrow K$ asserted in the problem description is true, the difference $\Pr(A = T) - \Pr(K = T)$ between the probability of the ace and the king is bounded by zero and one. It follows that $\Pr(A = T) \geq \Pr(K = T)$ and therefore that the ace is equally likely or more likely than the king, subject to the constraint $\Pr(P = T) = 1$ that the assertion in the problem statement is true.

7.2 The Monster from Paris

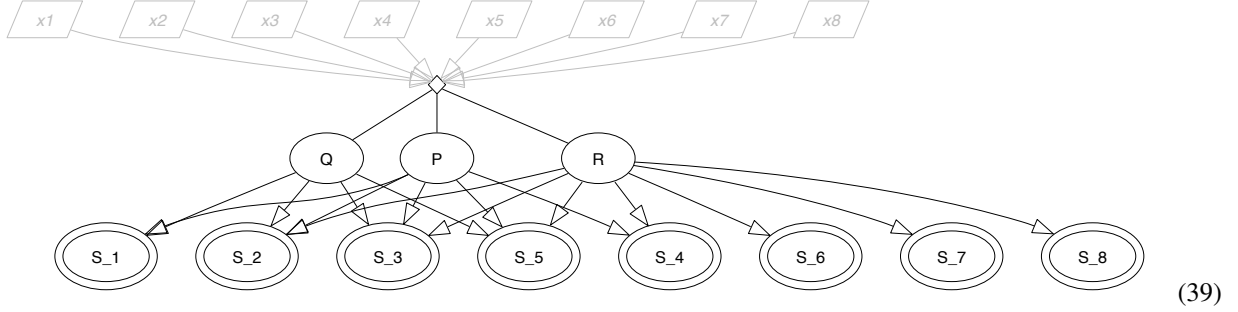
Next we examine a problem from Paris, Muiño, and Rosefield concerning inconsistent propositions [26]. We shall see that the parameterized consequence relation $\eta_{\triangleright \zeta}$ and the related concepts of maximal consistency and primary and secondary probability thresholds presented by these authors are simply indirect ways of describing linear optimization problems and their solutions. For this example there is a hypothetical creature about which there are three propositions: P that it is a chicken killer; Q that it is Japanese; and R that it is a salamander. There are also three compound formulas used as axiom-like assertions (here designated S_1 , S_2 and S_3):

$$S_1 := (P \wedge Q) \quad S_2 := (\neg(Q \wedge R) \wedge P) \quad S_3 := (R \wedge (\neg P \rightarrow (R \wedge Q))) \tag{37}$$

Additionally, there are several more compound formulas which are used as queries (here designated S_4 through S_8):

$$S_4 := (P \wedge R) \quad S_5 := (P \wedge (Q \vee R)) \quad S_6 := R \quad S_7 := \neg R \quad S_8 := (R \wedge \neg R) \tag{38}$$

The probability-network graph, which also includes the parameters x_1 through x_8 , is shown here:



For this parametric probability network we specify an uninformative probability distribution $\Pr_0(P, Q, R)$ on the primary variables P , Q , and R copied from the propositional variables, using the x_i parameters as in Section 3.1:

P	Q	R	$\Pr_0(P, Q, R)$
T	T	T	x_1
T	T	F	x_2
T	F	T	x_3
T	F	F	x_4
F	T	T	x_5
F	T	F	x_6
F	F	T	x_7
F	F	F	x_8

(40)

Each parameter x_i is subject to the constraint $0 \leq x_i \leq 1$ and they are collectively constrained by $\sum_i x_i = 1$. Following the method of Section 3.2 we construct a conditional probability table for each compound formula:

$\Pr_0(S_1 P, Q)$				$\Pr_0(S_2 P, Q, R)$					$\Pr_0(S_3 P, Q, R)$				
P	Q	$S_1 = T$	$S_1 = F$	P	Q	R	$S_2 = T$	$S_2 = F$	P	Q	R	$S_3 = T$	$S_3 = F$
T	T	1	0	T	T	T	0	1	T	T	T	1	0
T	F	0	1	T	T	F	1	0	T	T	F	0	1
F	T	0	1	T	F	T	1	0	T	F	T	1	0
F	F	0	1	T	F	F	1	0	T	F	F	0	1
				F	T	T	0	1	F	T	T	1	0
				F	T	F	0	1	F	T	F	0	1
				F	F	T	0	1	F	F	T	0	1
				F	F	F	0	1	F	F	F	0	1

(41)

$\Pr_0(S_4 P, R)$				$\Pr_0(S_5 P, Q, R)$				
P	R	$S_4 = T$	$S_4 = F$	P	Q	R	$S_5 = T$	$S_5 = F$
T	T	1	0	T	T	T	1	0
T	F	0	1	T	T	F	1	0
F	T	0	1	T	F	T	1	0
F	F	0	1	T	F	F	0	1
				F	T	T	0	1
				F	T	F	0	1
				F	F	T	0	1
				F	F	F	0	1

(42)

$\Pr_0(S_6 R)$		
R	$S_6 = T$	$S_6 = F$
T	1	0
F	0	1

$\Pr_0(S_7 R)$		
R	$S_7 = T$	$S_7 = F$
T	0	1
F	1	0

$\Pr_0(S_8 R)$		
R	$S_8 = T$	$S_8 = F$
T	0	1
F	0	1

(43)

Now let us ask, is it possible that the propositions $P \wedge Q$, $\neg(Q \wedge R) \wedge P$, and $R \wedge (\neg P \rightarrow (R \wedge Q))$ hold simultaneously? Since these are the compound formulas defined as S_1 , S_2 , and S_3 in Equation 37, the computed probability table $\Pr(S_1, S_2, S_3)$ provides the answer:

S_1	S_2	S_3	$\Pr(S_1, S_2, S_3)$
T	T	T	0
T	T	F	x_2
T	F	T	x_1
T	F	F	0
F	T	T	x_3
F	T	F	x_4
F	F	T	x_5
F	F	F	$x_6 + x_7 + x_8$

(44)

There are two impossible cases: that S_1 , S_2 , and S_3 hold simultaneously; and that S_1 holds but S_2 and S_3 do not.

Next let us address the matter of ‘maximal consistency’, which is a property defined by Paris *et al* to describe how compatible a set of logical formulas are with one another. According to their definition we seek the maximum threshold value such that the probability of each proposition in some set Γ attains at least that threshold; this maximum value is designated η . This definition of maximal consistency describes a linear optimization problem. In particular, the maximal consistency of the set $\{S_1, S_2, S_3\}$ of formulas from Equation 37 can be formulated as the linear optimization problem in which we ask for the maximum value of a new parameter z subject to the constraints $\Pr(S_1 = T) \geq z$, $\Pr(S_2 = T) \geq z$, and $\Pr(S_3 = T) \geq z$. This is straightforward to construct using parametric probability analysis. First we compute the marginal probability distribution on each compound formula in the set $\{S_1, S_2, S_3\}$ as in Section 4.2:

S_1	$\Pr(S_1)$
T	$x_1 + x_2$
F	$x_3 + x_4 + x_5 + x_6 + x_7 + x_8$

S_2	$\Pr(S_2)$
T	$x_2 + x_3 + x_4$
F	$x_1 + x_5 + x_6 + x_7 + x_8$

S_3	$\Pr(S_3)$
T	$x_1 + x_3 + x_5$
F	$x_2 + x_4 + x_6 + x_7 + x_8$

(45)

Then using the first element of each table in its respective constraint (along with the usual constraints to enforce the laws of probability) we construct the following optimization problem:

$$\begin{aligned}
&\text{maximize : } z \\
&\text{subject to : } x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 1 \\
&\quad x_1 + x_2 \geq z \\
&\quad x_2 + x_3 + x_4 \geq z \\
&\quad x_1 + x_3 + x_5 \geq z \\
&\text{and : } 0 \leq x_1 \leq 1 \\
&\quad 0 \leq x_2 \leq 1 \\
&\quad 0 \leq x_3 \leq 1 \\
&\quad 0 \leq x_4 \leq 1 \\
&\quad 0 \leq x_5 \leq 1 \\
&\quad 0 \leq x_6 \leq 1 \\
&\quad 0 \leq x_7 \leq 1 \\
&\quad 0 \leq x_8 \leq 1 \\
&\quad 0 \leq z \leq 1
\end{aligned}$$
(46)

Solving this linear program yields the maximum value $\eta = 0.667$ which is achieved at the following point:

$$(x_1 = 0.333, x_2 = 0.333, x_3 = 0.333, x_4 = 0.000, x_5 = 0.000, x_6 = 0.000, x_7 = 0.000, x_8 = 0.000, z = 0.667)$$
(47)

VARIABLE	FORMULA	THRESHOLD ζ
S_4	$P \wedge R$	0.667
S_5	$P \wedge (Q \vee R)$	1.000
S_6	R	0.667
S_7	$\neg R$	0.333
S_8	$R \wedge \neg R$	0.000

Table 5 Secondary threshold probabilities ζ for the formulas from Equation 38, using the maximal consistency $\eta = 0.667$ of the formulas $\{S_1, S_2, S_3\}$ from Equation 37 as the primary threshold probability.

This agrees with the result reported in [26]. Note that, although the author’s polynomial-optimization solver used floating-point arithmetic to compute the optimization results displayed here, there are exact rational solvers for linear programs which would instead calculate the solution precisely as the fraction $2/3$.

Moving on, we can formulate additional optimization problems to compute for other logical formulas the ‘secondary threshold probability’ designated ζ , when the solution value η above is used as the ‘primary threshold probability’. By the definition in [26] we seek the maximum probability ζ of each queried formula subject to the constraints that every formula in the designated set Γ must attain probability at least η . Again this definition describes certain linear optimization problems. For this example, to find secondary threshold probabilities of the formulas defined as S_4 through S_8 in Equation 38 we first calculate their symbolic polynomial probabilities:

$$\begin{array}{|c|c|} \hline S_4 & \Pr(S_4) \\ \hline \hline T & x_1 + x_3 \\ \hline F & x_2 + x_4 + x_5 + x_6 + x_7 + x_8 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline S_5 & \Pr(S_5) \\ \hline \hline T & x_1 + x_2 + x_3 \\ \hline F & x_4 + x_5 + x_6 + x_7 + x_8 \\ \hline \end{array} \quad (48)$$

$$\begin{array}{|c|c|} \hline S_6 & \Pr(S_6) \\ \hline \hline T & x_1 + x_3 + x_5 + x_7 \\ \hline F & x_2 + x_4 + x_6 + x_8 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline S_7 & \Pr(S_7) \\ \hline \hline T & x_2 + x_4 + x_6 + x_8 \\ \hline F & x_1 + x_3 + x_5 + x_7 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline S_8 & \Pr(S_8) \\ \hline \hline T & 0 \\ \hline F & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 \\ \hline \end{array} \quad (49)$$

Now to compute the secondary threshold probability for the query $S_4 := (P \wedge R)$ relative to the set $\{S_1, S_2, S_3\}$ using the primary threshold probability η , we query the maximum value of $\Pr(S_4 = T)$ subject to the constraints $\Pr(S_1 = T) \geq z$, $\Pr(S_2 = T) \geq z$, and $\Pr(S_3 = T) \geq z$ where the auxiliary parameter z is now fixed at the desired primary threshold η . Using the solution $\eta = 0.667$ from the optimization problem in Equation 46 as the primary threshold probability, we construct this optimization problem to compute the secondary threshold probability for the formula $S_4 := (P \wedge R)$:

$$\begin{aligned}
&\text{maximize : } x_1 + x_3 \\
&\text{subject to : } x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 1 \\
&\quad x_1 + x_2 \geq z \\
&\quad x_2 + x_3 + x_4 \geq z \\
&\quad x_1 + x_3 + x_5 \geq z \\
&\text{and : } 0 \leq x_1 \leq 1 \\
&\quad 0 \leq x_2 \leq 1 \\
&\quad 0 \leq x_3 \leq 1 \\
&\quad 0 \leq x_4 \leq 1 \\
&\quad 0 \leq x_5 \leq 1 \\
&\quad 0 \leq x_6 \leq 1 \\
&\quad 0 \leq x_7 \leq 1 \\
&\quad 0 \leq x_8 \leq 1 \\
&\quad z = 0.667
\end{aligned} \quad (50)$$

Solving this linear program yields the maximum value $\zeta = 0.667$ for the queried formula $S_4 := (P \wedge R)$. By similar analysis we compute secondary threshold probabilities for the remaining formulas in Equation 38, with the results displayed in Table 5. Notably for the query formula R the secondary threshold $\zeta = 1/3$ was reported in [26]; but the maximal solution to the optimization problem suggested by the text is twice this value.

7.3 Goodman's Hot Buttered Conditionals

Next let us visit Goodman's treatment of counterfactual conditional statements using his opening example from *Fact, Fiction, and Forecast* [11]. In his own words:

What, then, is the *problem* about counterfactual conditionals? Let us confine ourselves to those in which antecedent and consequent are inalterably false—as, for example, when I say of a piece of butter that was eaten yesterday, and that had never been heated,

If that piece of butter had been heated to 150° F., it would have melted.

Considered as truth-functional compounds, all counterfactuals are of course true, since their antecedents are false. Hence

If that piece of butter had been heated to 150° F., it would not have melted

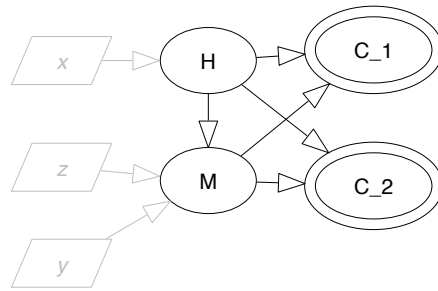
would also hold. Obviously something different is intended, and the problem is to define the circumstances under which a given counterfactual holds while the opposing counterfactual with the contradictory consequent fails to hold.

Let us use H to represent the proposition that the butter had been heated, and M for the proposition that it melted. Each variable has the set $\{T, F\}$ of possible elementary values representing truth and falsity. It is indeed the case that given the axiom that H is false, both statements of material implication $H \rightarrow M$ and $H \rightarrow \neg M$ are true. In fact, Boolean polynomial representation shows that the conjunction $(H \rightarrow M) \wedge (H \rightarrow \neg M)$ simplifies to the negation $\neg H$. In other words, interpreting them as statements of material implication, the logical conjunction of the two opposing conditional statements above is equivalent to the single unconditional statement:

That piece of butter had not been heated to 150° F.

If it desired to meet Goodman's dichotomy criterion (that one of the conditional statements holds but the opposing statement with the contradictory consequent does not hold) then it is not wise to model his conditional sentences as statements of material implication.

Parametric probability networks and direct probability encoding provide the desired 'something different' to model conditional and counterfactual statements. The resulting models and analysis meet Goodman's dichotomy criterion and give otherwise reasonable results. To illustrate, let us build a parametric probability network including the primary variables H and M to represent the propositional variables, along with the primary variables $C_1 := (H \rightarrow M)$ and $C_2 := (H \rightarrow \neg M)$ to represent the corresponding statements of material implication; we add parameters x , y , and z . Here is the network graph:



(51)

First we specify an uninformative prior distribution on H and M using parametric distributions $\Pr_0(H)$ and $\Pr_0(M|H)$, with constraints $0 \leq x \leq 1$, $0 \leq y \leq 1$, and $0 \leq z \leq 1$:

H	$\Pr_0(H)$
T	x
F	$1 - x$

$\Pr_0(M H)$		
H	$M = T$	$M = F$
T	y	$1 - y$
F	z	$1 - z$

(52)

Let us interpret Goodman's first sentence as the constraint $\Pr_0(M = T | H = T) = 1$ that the input value of the conditional probability that M is true given that H is true is one; likewise let us interpret his second sentence as the constraint $\Pr_0(M = T | H = T) = 0$ or the equivalent $\Pr_0(M = F | H = T) = 1$. Now it is a matter of elementary algebra that these statements are inconsistent: the first says $y = 1$ and the others say $y = 0$. This algebraic inconsistency does not involve the parameter x that encodes the prior probability $\Pr_0(H = T)$ that the butter had been heated. In other words, these opposing conditional sentences are inconsistent specifically because their consequents disagree and not because they are counterfactual. Additionally, in this interpretation these conditional sentences do not have existential import. Neither constraint $y = 1$ nor $y = 0$ requires that H must certainly be true ($x = 1$), nor even that H must possibly be true ($x > 0$); these constraints on y do not affect x at all.

Next let us compute the conditional probability $\Pr(M | H)$ of whether the butter melted given whether it had been heated, alongside the marginal probability $\Pr(M)$ of whether the butter melted (integrating the cases that it had or had not been heated) and the marginal probability $\Pr(H)$ of whether the butter had been heated:

$\Pr(M H)$		
H	$M = T$	$M = F$
T	$(xy) / (x)$	$(x - xy) / (x)$
F	$(z - xz) / (1 - x)$	$(1 - x - z + xz) / (1 - x)$

M	$\Pr(M)$
T	$z + xy - xz$
F	$1 - z - xy + xz$

H	$\Pr(H)$
T	x
F	$1 - x$

(53)

These computed probability tables already tell an interesting story about Goodman's conditional sentences. First, the output probability $\Pr(H = T)$ that the butter had been heated has exactly the same value as the input probability $\Pr_0(H = T)$, namely the parameter x . However the output probability $\Pr(M = T | H = T)$ differs in a subtle way from the input probability $\Pr_0(M = T | H = T)$; the input value is the parameter y but the output value is the quotient xy/x . Therefore if it is certain that the butter had not been heated ($x = 0$) then the computed conditional probability xy/x that the butter melted given this now-impossible condition is algebraically indeterminate: it is the quotient $0/0$.

Notwithstanding this exceptional *conditional* probability, the overall *marginal* probability that the butter melted subject to the constraint $x = 0$ that the butter certainly had not been heated does not involve division by zero; in fact it does not involve division at all. The computed probability $\Pr(M = T)$, whose value is the polynomial $z + xy - xz$, simplifies to z when $x = 0$. In other words, subject to the constraint that the butter certainly had not been heated, the probability that the butter melted is exactly the value z specified as the input component probability $\Pr_0(M = T | H = F)$. If the user has not provided any more information about z then its precise value remains indeterminate; it is only constrained by $0 \leq z \leq 1$ to satisfy the laws of probability. In the terminology of Section 6.1, we have just considered subjunctive and imperative modes of asking the same question: What is the probability that M is true, given the condition that H is false? In either formulation the answer is the same: The queried probability is indeterminate.

We have just analyzed Goodman's counterfactual conditional sentences using parametric probability directly, without intermediate formulas from the propositional calculus. However it is instructive to embed the formulas $H \rightarrow M$ and $H \rightarrow \neg M$ into our probability network and to compute the probabilities associated with them. Using the definitions $C_1 := (H \rightarrow M)$ and $C_2 := (H \rightarrow \neg M)$ and the method of Section 3.2 we construct the following conditional probability tables (here labeled with the embedded formulas instead of with the primary-variable names C_1 and C_2):

$\Pr_0((H \rightarrow M) H, M)$			
H	M	$(H \rightarrow M) = T$	$(H \rightarrow M) = F$
T	T	1	0
T	F	0	1
F	T	1	0
F	F	1	0

$\Pr_0((H \rightarrow \neg M) H, M)$			
H	M	$(H \rightarrow \neg M) = T$	$(H \rightarrow \neg M) = F$
T	T	0	1
T	F	1	0
F	T	1	0
F	F	1	0

(54)

We compute the marginal probabilities that each embedded formula is true:

$(H \rightarrow M)$	$\Pr((H \rightarrow M))$
T	$1 - x + xy$
F	$x - xy$

$(H \rightarrow \neg M)$	$\Pr((H \rightarrow \neg M))$
T	$1 - xy$
F	xy

(55)

Solving either equation $1 - x + xy = 1 - xy$ or $x - xy = xy$ reveals that there are exactly two cases in which the opposing statements of material implication have the same probability of truth: when $x = 0$ (in which case each statement is true

with certainty) and when $y = \frac{1}{2}$ (in which case each statement is true with probability $1 - \frac{1}{2}x$). In other words, if it is certain *a priori* that the butter had not been heated ($x = 0$) then both formulas $H \rightarrow M$ and $H \rightarrow \neg M$ must certainly be true. However if there is exactly a 50% chance that the butter melted after it had been heated ($y = \frac{1}{2}$) then it is also equally likely that the formulas $H \rightarrow M$ and $H \rightarrow \neg M$ are true; but now their mutual probability ($1 - \frac{1}{2}x$) is one minus half the prior probability that the butter had been heated.

The relation that both formulas $H \rightarrow M$ and $H \rightarrow \neg M$ are *equally likely* to be true is different from the relation that both formulas are *simultaneously* true; both relations occur when $x = 0$ but only the former occurs when $y = \frac{1}{2}$ and $x \neq 0$. Evaluating the joint probability $\Pr(H, C_1, C_2)$ of H and both statements of material implication provides additional detail that is hidden in the marginal probabilities above:

INDEX	H	$(H \rightarrow M)$	$(H \rightarrow \neg M)$	$\Pr(H, (H \rightarrow M), (H \rightarrow \neg M))$
1	T	T	T	0
2	T	T	F	xy
3	T	F	T	$x - xy$
4	T	F	F	0
5	F	T	T	$1 - x$
6	F	T	F	0
7	F	F	T	0
8	F	F	F	0

(56)

This probability table shows that it is impossible that the antecedent H and both opposing statements of material implication are simultaneously true (the probability of this event, which appears in row 1 of the table, is identically zero). Moreover when H is true exactly one of the opposing statements of material implication must hold (see rows 1 through 4). But when H is false both statements of material implication must be true (see rows 5 through 8). To recover the probabilities in Equation 55 from the probabilities in Equation 56 it is necessary to add appropriate elements of the latter probability table. For example $\Pr((H \rightarrow M) = T)$ is given by the sum $(0) + (xy) + (1 - x) + (0)$ of the polynomials from rows 1, 2, 5, and 6, which yields $1 - x + xy$.

7.4 Smullyan's Knights, Knaves, and Zombies

Finally we consider two of Smullyan's problems from *What Is the Name of This Book?* which were also analyzed by Kolany using a rather different technique [30, 17]. First a basic knights and knaves problem for which we must simply answer a probability-table query; and second a zombie problem in which we must first answer a probability-table query and then perform a search to find certain values of the parameters in the resulting probability table.

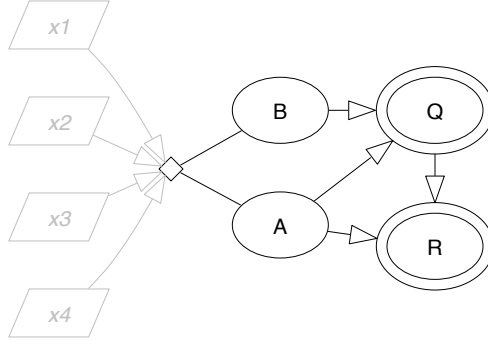
WE ARE THE KNIGHTS WHO SAY... The background for the first problem (number 36 in [30]) is that on the imagined island, knights always tell the truth and knaves always lie.

Once when I visited the island of knights and knaves, I came across two of the inhabitants resting under a tree. I asked one of them, "Is either of you a knight?" He responded, and I knew the answer to my question.

What is the person to whom I addressed the question—is he a knight or knave; And what is the other one?

In the parametric probability model let us use the variable A to represent the proposition that the respondent is a knight, and B for the proposition that the other inhabitant is a knight. We define $Q := (A \vee B)$ to represent the true answer to the question of whether either inhabitant is a knight. We introduce R to represent the response that is given: R is true if the response is 'yes' and false if the response is 'no'. By Smullyan's rules for the island, if Q were true then the inhabitant would respond 'yes' if and only if he were a knight; and if Q were false then the responses would be opposite. Hence the definition $R := (A \leftrightarrow Q)$ using the biconditional. Here is the network graph, which also includes

the parameters x_1 through x_4 :



(57)

We specify an uninformative parametric probability distribution on A and B using the x_i parameters with the usual constraints $0 \leq x_i \leq 1$ and $\sum_i x_i = 1$ as in Section 3.1; we construct the appropriate component probability tables for Q and R according to Section 3.2:

A	B	$\Pr_0(A, B)$	$\Pr_0(Q A, B)$				$\Pr_0(R Q, A)$				(58)
A	B		$Q = \text{T}$	$Q = \text{F}$	Q	A	$R = \text{T}$	$R = \text{F}$			
T	T	x_1	T	T	1	0	T	T	1	0	
T	F	x_2	T	F	1	0	T	F	0	1	
F	T	x_3	F	T	1	0	F	T	0	1	
F	F	x_4	F	F	0	1	F	F	1	0	

(58)

We compute the joint probability of the identities A and B conditioned on the response R :

$\Pr(A, B R)$				
R	$A = T, B = T$	$A = T, B = F$	$A = F, B = T$	$A = F, B = F$
T	$(x_1)/(x_1 + x_2 + x_4)$	$(x_2)/(x_1 + x_2 + x_4)$	$(0)/(x_1 + x_2 + x_4)$	$(x_4)/(x_1 + x_2 + x_4)$
F	$(0)/(x_3)$	$(0)/(x_3)$	$(x_3)/(x_3)$	$(0)/(x_3)$

(59)

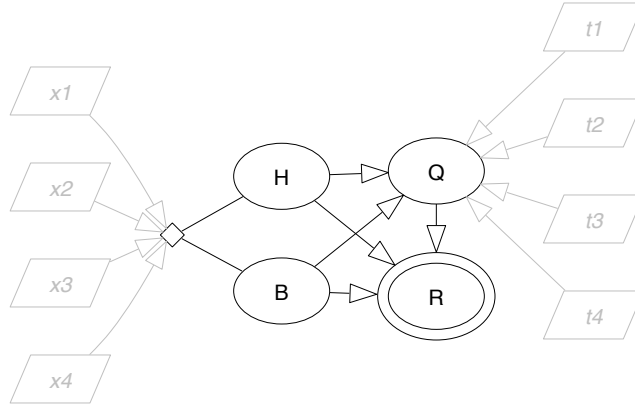
This result table for $\Pr(A, B|R)$ gives the solution. An answer of ‘yes’ ($R = T$) leaves three possible configurations of identities, but an answer of ‘no’ ($R = F$) leaves only one possible configuration of identities: that the responder is a knave and the other inhabitant is a knight ($A = F, B = T$). Therefore if the identities of the inhabitants are known with certainty after the response, then the response must have been ‘no’, the responder must be a knave, and his fellow inhabitant must be a knight.

ZOMBIELAND For the second problem (number 160 in [30]) we visit Smullyan’s island of zombies. The custom here is that zombies always lie and humans always tell the truth. However instead of ‘yes’ and ‘no’ the inhabitants answer questions with ‘Bal’ or ‘Da’; one means ‘yes’ and the other means ‘no’ but we do not know which is which. The problem asks the following:

Suppose you are not interested in what “Bal” means, but only in whether the speaker is a zombie. How can you find this out in only one question? (Again, he will answer “Bal” or “Da.”)

For the parametric probability model of this problem we use the variable H to represent whether the speaker is human ($H = T$) or zombie; B to represent whether ‘Bal’ means ‘yes’ ($B = T$) or ‘no’; Q for the true answer to the unknown question that is sought; and R for whether the speaker gives the response ‘Bal’ ($R = T$) or ‘Da’. Here is the network

graph:



(60)

The prior distribution $\Pr_0(H, B)$ on H and B is specified using parameters x_1 through x_4 with $0 \leq x_i \leq 1$ and $\sum_i x_i = 1$. Additional parameters t_1 through t_4 with each $t_i \in \{0, 1\}$ are introduced as described in Section 3.3 to specify the component probability table $\Pr_0(Q|H, B)$ of the unknown question Q . Here are both component probability tables:

H	B	$\Pr_0(H, B)$
T	T	x_1
T	F	x_2
F	T	x_3
F	F	x_4

$\Pr_0(Q H, B)$			
H	B	$Q = T$	$Q = F$
T	T	t_1	$1 - t_1$
T	F	t_2	$1 - t_2$
F	T	t_3	$1 - t_3$
F	F	t_4	$1 - t_4$

(61)

The speaker's response R can be modeled in the propositional calculus using nested biconditionals to represent the rules of the island. The formula $H \leftrightarrow Q$ reveals whether the speaker will answer in the affirmative. For example a zombie ($H = F$) will dishonestly provide an affirmative answer to a question that is actually false ($Q = F$) but a human will honestly provide a negative answer to a question that is actually false. Relating this inner biconditional to whether 'Bal' means 'yes' using another biconditional $(H \leftrightarrow Q) \leftrightarrow B$ then reveals whether the speaker will answer 'Bal' (if this outer biconditional is true) or 'Da'. For example, if the speaker will answer in the affirmative, then he will respond with 'Bal' if and only if that means 'yes'. The following component probability table $\Pr_0(R|H, Q, B)$ constructed as in Section 3.2 implements the definition $R := ((H \leftrightarrow Q) \leftrightarrow B)$:

$\Pr_0(R H, Q, B)$				
H	Q	B	$R = T$	$R = F$
T	T	T	1	0
T	T	F	0	1
T	F	T	0	1
T	F	F	1	0
F	T	T	0	1
F	T	F	1	0
F	F	T	1	0
F	F	F	0	1

(62)

It takes two phases of analysis to find a question that will determine whether the speaker is human or zombie. For the primary analysis we compute the joint probability $\Pr(R, H)$ of each identity and each response using symbolic

(a)

INDEX	t_1	t_2	t_3	t_4	$\Pr(R = T, H = T)$
1	0	0	0	0	x_2
2	0	0	0	1	x_2
3	0	0	1	0	x_2
4	0	0	1	1	x_2
5	0	1	0	0	0
6	0	1	0	1	0
7	0	1	1	0	0
8	0	1	1	1	0
9	1	0	0	0	$x_1 + x_2$
10	1	0	0	1	$x_1 + x_2$
11	1	0	1	0	$x_1 + x_2$
12	1	0	1	1	$x_1 + x_2$
13	1	1	0	0	x_1
14	1	1	0	1	x_1
15	1	1	1	0	x_1
16	1	1	1	1	x_1

(c)

INDEX	t_1	t_2	t_3	t_4	$\Pr(R = F, H = T)$
1	0	0	0	0	x_1
2	0	0	0	1	x_1
3	0	0	1	0	x_1
4	0	0	1	1	x_1
5	0	1	0	0	$x_1 + x_2$
6	0	1	0	1	$x_1 + x_2$
7	0	1	1	0	$x_1 + x_2$
8	0	1	1	1	$x_1 + x_2$
9	1	0	0	0	0
10	1	0	0	1	0
11	1	0	1	0	0
12	1	0	1	1	0
13	1	1	0	0	x_2
14	1	1	0	1	x_2
15	1	1	1	0	x_2
16	1	1	1	1	x_2

(b)

INDEX	t_1	t_2	t_3	t_4	$\Pr(R = T, H = F)$
1	0	0	0	0	x_3
2	0	0	0	1	$x_3 + x_4$
3	0	0	1	0	0
4	0	0	1	1	x_4
5	0	1	0	0	x_3
6	0	1	0	1	$x_3 + x_4$
7	0	1	1	0	0
8	0	1	1	1	x_4
9	1	0	0	0	x_3
10	1	0	0	1	$x_3 + x_4$
11	1	0	1	0	0
12	1	0	1	1	x_4
13	1	1	0	0	x_3
14	1	1	0	1	$x_3 + x_4$
15	1	1	1	0	0
16	1	1	1	1	x_4

(d)

INDEX	t_1	t_2	t_3	t_4	$\Pr(R = F, H = F)$
1	0	0	0	0	x_4
2	0	0	0	1	0
3	0	0	1	0	$x_3 + x_4$
4	0	0	1	1	x_3
5	0	1	0	0	x_4
6	0	1	0	1	0
7	0	1	1	0	$x_3 + x_4$
8	0	1	1	1	x_3
9	1	0	0	0	x_4
10	1	0	0	1	0
11	1	0	1	0	$x_3 + x_4$
12	1	0	1	1	x_3
13	1	1	0	0	x_4
14	1	1	0	1	0
15	1	1	1	0	$x_3 + x_4$
16	1	1	1	1	x_3

Table 6 Secondary analysis to distinguish humans from zombies, using the polynomials in the result table for $\Pr(R, H)$ shown in Equation 63 instantiated at different values of the parameters t_1 through t_4 . We search for values of (t_1, t_2, t_3, t_4) such that exactly one of $\Pr(R = T, H = T)$ or $\Pr(R = T, H = F)$ is zero, and exactly one of $\Pr(R = F, H = T)$ or $\Pr(R = F, H = F)$ is zero. The values at rows 6 and 11 meet these criteria.

probability-network inference as in Section 4.2:

INDEX	R	H	$\Pr(R, H)$
1	T	T	$x_2 + t_1x_1 - t_2x_2$
2	T	F	$x_3 - t_3x_3 + t_4x_4$
3	F	T	$x_1 - t_1x_1 + t_2x_2$
4	F	F	$x_4 + t_3x_3 - t_4x_4$

(63)

For the secondary analysis we must find values of the parameters (t_1, t_2, t_3, t_4) such that the question Q encoded by those values successfully discriminates between humans and zombies. With regard to the result table for $\Pr(R, H)$ in Equation 63, successful discrimination requires that after substitution of the chosen values of the t_i parameters, exactly one of the first two elements of $\Pr(R, H)$ is identically zero and that exactly one of the second two elements is identically zero. As in Section 5.3 we can set up a simple exhaustive search to find such values by substituting each of the sixteen possible values of (t_1, t_2, t_3, t_4) with each $t_i \in \{0, 1\}$ into each of the four polynomials in Equation 63. Table 6 shows these substituted polynomial values. There are two vectors of parameter values that meet the search criteria: $(0, 1, 0, 1)$ at row 6 and $(1, 0, 1, 0)$ at row 11. With reference to Equation 61, here is the question Q instantiated using the second solution $(t_1, t_2, t_3, t_4) = (1, 0, 1, 0)$:

$\Pr_0(Q H, B)$			
H	B	$Q = T$	$Q = F$
T	T	1	0
T	F	0	1
F	T	1	0
F	F	0	1

(64)

Using this solution the primary variable Q expresses the logical formula B representing whether ‘Bal’ means ‘yes’; in other words $Q := B$. In this case the joint probability of identity and response, obtained by substituting the selected parameter values into Equation 63, is given by:

INDEX	R	H	$\Pr(R, H)$
1	T	T	$x_1 + x_2$
2	T	F	0
3	F	T	0
4	F	F	$x_3 + x_4$

(65)

In other words, the question “Does ‘Bal’ mean ‘yes’?” will reliably distinguish a human speaker from a zombie: a human ($H = T$) must answer ‘Bal’ ($R = T$) but a zombie must answer ‘Da’. The other solution $(0, 1, 0, 1)$ for the parameters (t_1, t_2, t_3, t_4) gives the negation of this question which works just as well; in response to the question “Does ‘Da’ mean ‘yes’?” (corresponding to the formula $\neg B$) a human must answer ‘Da’ but a zombie must answer ‘Bal’. Notably, Kolany incorrectly matched the question from the first solution with the responses from the second solution: “... we could ask him whether *Bal* means *Yes*. If he answers *Bal*, he is a zombie.” [17]

Note that certain prior assumptions about whether the speaker is human versus zombie will lead to exceptions. Here are the computed probability distributions $\Pr(H)$ and $\Pr(R)$:

H	$\Pr(H)$	R	$\Pr(R)$
T	$x_1 + x_2$	T	$x_1 + x_2$
F	$x_3 + x_4$	F	$x_3 + x_4$

(66)

Had it been specified *a priori* that there were no humans on the island (with the constraint $x_1 + x_2 = 0$) then it would be impossible under the rules of the island for the speaker to answer ‘Bal’; in other words $\Pr(R = T)$, which has the polynomial value $x_1 + x_2$, would also be constrained to equal zero. Likewise had it been specified that there were no zombies on the island ($x_3 + x_4 = 0$) then it would be impossible for the speaker to answer ‘Da’ ($R = F$).

8 CONCLUSION

The method of *parametric probability analysis*, introduced and illustrated above, demonstrates that probability and classical logic are not only compatible but also complementary. To adopt a popular turn of phrase, there is no daylight between logic and probability. Many so-called ‘logic’ problems are more specifically probability problems, because they require reasoning about the probabilities of formulas from the propositional calculus. For such problems it is useful to embed logical formulas inside parametric probability networks. Many other logic problems are better represented directly as parametric probability networks, without use of the propositional calculus or of first-order logic at all. Parametric probability analysis complements classical logic by providing a powerful set of computational tools for modeling and reasoning about implication, consequence, and quantification.

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Implementation note: pqlsh-8.0.7+cplex+openmp 2012-05-23

A PROBABILITY MODEL SOURCE CODE

A.1 Ace-King

```
// ace-king.pql: Johnson-Laird Acta Psych 1996 via Bringsford J Applied Logic 2008

primary A { label = "There is an ace in the hand"; states = binary; }
primary K { label = "There is a king in the hand"; states = binary; }
clique _C; potential ( _C : A K ) { parametric(x); }

primary P {
  label = "Value of  $((K \rightarrow A) \rightarrow K)$ ";
  states = binary;
}
probability ( P | A K ) { function = ( "P <-> ((K <-> A) <-> K) ? 1 : 0" ); }
```

A.2 Amphibian

```
// amphibian.pql: adopted from Paris, Muino, Rosefield 2009

primary P { label = "Chicken killer"; states = binary; }
primary Q { label = "Japanese"; states = binary; }
primary R { label = "Salamander"; states = binary; }
// fully parametric prior distribution, no independence assumptions
clique _C; probability ( _C : P Q R ) { parametric(x); }

// beliefs (like axioms)

primary S_1 { label = "Value of  $(P \wedge Q)$ "; states = binary; }
probability ( S_1 | P Q ) { function = ( "S_1 <-> P && Q ? 1 : 0" ); }

primary S_2 { label = "Value of  $(\neg (Q \wedge R) \wedge P)$ "; states = binary; }
probability ( S_2 | P Q R ) { function = ( "S_2 <-> !(Q && R) && P ? 1 : 0" ); }

primary S_3 { label = " $S_3 :: R \wedge (\neg P \rightarrow (R \wedge Q))$ "; states = binary; }
probability ( S_3 | P Q R ) { function = ( "S_3 <-> R && (!P -> (R && Q)) ? 1 : 0" ); }

// queries

primary S_4 { label = "Value of  $(S_4 :: P \wedge R)$ "; states = binary; }
probability ( S_4 | P R ) { function = ( "S_4 <-> P && R ? 1 : 0" ); }

primary S_5 { label = "Value of  $(P \wedge (Q \vee R))$ "; states = binary; }
probability ( S_5 | P Q R ) { function = ( "S_5 <-> P && (Q || R) ? 1 : 0" ); }

primary S_6 { label = "Value  $(R)$ "; states = binary; }
probability ( S_6 | R ) { function = ( "S_6 <-> R ? 1 : 0" ); }

primary S_7 { label = "Value of  $(\neg R)$ "; states = binary; }
probability ( S_7 | R ) { function = ( "S_7 <-> !R ? 1 : 0" ); }

primary S_8 { label = "Value of  $(R \wedge \neg R)$ "; states = binary; }
probability ( S_8 | R ) { function = ( "S_8 <-> R && !R ? 1 : 0" ); }

net { graph = "rankdir = TB;"; }
net { graph = 'subgraph { rank=same; "P"; "Q"; "R"}'; }
```

A.3 Counterfactual Conditions

```
// butter.pql: Goodman's counterfactual from Fact, Fiction, and Forecast

parameter x { range = (0,1); }
parameter y { range = (0,1); }
parameter z { range = (0,1); }

primary H { label = "The butter was heated"; states = binary; }
probability ( H ) { data = (x, 1-x); }

primary M { label = "The butter melted"; states = binary; }
probability ( M | H ) { data = (y, 1-y, z, 1-z); }

primary C_1 {
  label = "Value of  $(H \rightarrow M)$ "; tex = "(H \rightarrow M)";
  states = binary;
}
probability ( C_1 | H M ) { function = "C_1 <-> H -> M ? 1 : 0"; }

primary C_2 {
  label = "Value of  $(H \rightarrow \neg M)$ "; tex = "(H \rightarrow \neg M)";
  states = binary;
}
probability ( C_2 | H M ) { function = "C_2 <-> H -> !M ? 1 : 0"; }

net { graph = 'subgraph { rank=same; "H"; "M"; }'; }
```

A.4 Knight or Knave

```
// knight2.pql: Kolany's example 2, from Smullyan 1978 #36 p. 23
// I asked one of them, "Is either of you a knight?"

primary A { label = "A is a knight"; states = binary; }
primary B { label = "B is a knight"; states = binary; }
clique _C; probability ( _C : A B ) { parametric(x); }

primary Q { label = "Question:  $A \vee B$ "; states = binary; }
probability ( Q | A B ) { function = ( "(Q <-> A || B) ? 1 : 0" ); }

primary R { label = "A's response:  $A \leftrightarrow Q$ "; states = binary; }
probability ( R | Q A ) { function = ( "R <-> (Q <-> A) ? 1 : 0" ); }

net { graph = 'subgraph { rank=same; "Q"; "R"; }'; }
```

A.5 Human or Zombie: Primary Analysis

```
// zombie1.pql: Kolany's example 3 from Smullyan 1978 #160 p. 150

primary H { label = "Speaker is human"; states = binary; }
primary B { label = "'Bal' means 'yes'"; states = binary; }
primary Q { label = "Question (parametric in  $t_i$ )"; states = binary; }
probability ( Q | H B ) { parametric(t); }
clique _C; probability ( _C : H B ) { parametric(x); }

primary R {
```

```

    label = "Response is 'Bal':  $((H \rightarrow Q) \rightarrow B)$ ";
    states = binary;
}
probability ( R | H Q B ) { function = ( "R <-> ((H <-> Q) <-> B) ? 1 : 0" ); }

net { graph = 'subgraph { rank=same; "Q"; "R"; }'; }
net { graph = 'subgraph { rank=max; "t1"; "t2"; "t3"; "t4"; }'; }

```

A.6 Human or Zombie: Secondary Analysis

```

// zombie1-search.pql: Kolany's example 3 from Smullyan 1978 #160 p. 150

decision t[1] { states = values(0,1); }
decision t[2] { states = values(0,1); }
decision t[3] { states = values(0,1); }
decision t[4] { states = values(0,1); }

// sequence of decisions; not important here
probability ( t[1] ) {}
probability ( t[2] | t[1] ) {}
probability ( t[3] | t[2] ) {}
probability ( t[4] | t[3] ) {}

// to allow substitution of t[i] values; not really decision-theoretic utilities
utility U_1 { tex = "\prob{R=\state{T},H=\state{T}}"; range = (0,1); }
utility U_2 { tex = "\prob{R=\state{T},H=\state{F}}"; range = (0,1); }
utility U_3 { tex = "\prob{R=\state{F},H=\state{T}}"; range = (0,1); }
utility U_4 { tex = "\prob{R=\state{F},H=\state{F}}"; range = (0,1); }

// polynomials are from the result Pr( R, H ) using zombie1.pql
probability ( U_1 | t[1] t[2] t[3] t[4] ) { function = "x2 + t1*x1 - t2*x2"; }
probability ( U_2 | t[1] t[2] t[3] t[4] ) { function = "x3 - t3*x3 + t4*x4"; }
probability ( U_3 | t[1] t[2] t[3] t[4] ) { function = "x1 - t1*x1 + t2*x2"; }
probability ( U_4 | t[1] t[2] t[3] t[4] ) { function = "x4 + t3*x3 - t4*x4"; }

primary H { label = "Speaker is human"; states = binary; }
primary B { label = "'Bal' means 'yes'"; states = binary; }

// Question 11: Does 'Bal' mean 'yes'?
set t[1] = 1; set t[2] = 0; set t[3] = 1; set t[4] = 0;

primary Q { label = "Question (parametric in $t_i$)"; states = binary; }
probability ( Q | H B ) { parametric(t); }
clique _C; probability ( _C : H B ) { parametric(x); }

primary R {
    label = "Response is 'Bal': value of  $(Q \rightarrow (H \rightarrow B))$ ";
    states = binary;
}
probability ( R | Q H B ) { function = ( "R <-> (Q <-> (H <-> B)) ? 1 : 0" ); }

net { graph = 'subgraph { rank=same; "Q"; "R"; }'; }
net { graph = 'subgraph { rank=max; "t1"; "t2"; "t3"; "t4"; }'; }

```

CONTENTS

1	Introduction	1
2	Parametric Probability Networks	2
2.1	The Problem with Penguins	2
2.2	Variables and Constraints	2
2.3	Network Graph and Component Probability Tables	3
2.4	Computable Specification	4
3	Embedded Mathematical Systems	5
3.1	Prior Probabilities for Propositional Variables	5
3.2	Conditional Probabilities for Operations	6
3.3	Known and Unknown Unknowns	7
4	Primary Analysis: Symbolic Probability Inference	8
4.1	Structured Probability Queries	8
4.2	Simple Table-Based Probability Inference	9
4.3	Linear Functions Follow Form	11
4.4	Handling Division by Zero	12
4.5	Boolean Polynomials and Coincident Probabilities	13
5	Secondary Analysis: Algebra, Optimization, and Search	14
5.1	Algebra with Polynomials	14
5.2	Polynomial Optimization	15
5.3	General Search	16
6	Additional Modeling Issues	17
6.1	Subjunctive Conditions, Imperative Constraints	17
6.2	Direct Probability Encoding	19
7	Analysis of Selected Problems	20
7.1	Johnson-Laird’s Winning Hand	20
7.2	The Monster from Paris	23
7.3	Goodman’s Hot Buttered Conditionals	27
7.4	Smullyan’s Knights, Knaves, and Zombies	29
8	Conclusion	34
A	Probability Model Source Code	36
A.1	Ace-King	36
A.2	Amphibian	36
A.3	Counterfactual Conditions	37
A.4	Knight or Knave	37
A.5	Human or Zombie: Primary Analysis	37
A.6	Human or Zombie: Secondary Analysis	38